Integration of the Finite Toda Lattice with Complex-Valued Initial Data

Aydin Huseynov* and Gusein Sh. Guseinov**

**Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, AZ1141 Baku, Azerbaijan (e-mail: huseynov@email.com)*

***Department of Mathematics, Atilim University, 06836 Incek, Ankara, Turkey (e-mail: guseinov@atilim.edu.tr)*

Abstract: We show that the finite Toda lattice with complex-valued initial data can be integrated by the inverse spectral method. For this goal spectral data for complex Jacobi matrices are introduced and an inverse spectral problem from the spectral data is solved. The time evolution of the spectral data for the Jacobi matrix associated with the solution of the Toda lattice is computed. Using the solution of the inverse spectral problem with respect to the time-dependent spectral data we reconstruct the time-dependent Jacobi matrix and hence the desired solution of the finite Toda lattice.

Keywords: Toda lattice, Jacobi matrix, difference equation, spectral data, inverse spectral problem.

1. INTRODUCTION

There are three kinds of the Toda lattices; finite, semi-infinite, and doubly-infinite Toda lattices. There is a huge number of papers devoted to the investigation of the Toda lattices and their various generalizations, from which we indicate here only [1–12]. For some techniques of investigation of nonlinear discrete equations (difference equations) see [13–15]. In the present paper, we deal with the finite Toda lattice subject to the complex-valued initial conditions.

The (open) *finite Toda lattice* is a Hamiltonian system which describes the motion of N particles moving in a straight line, with "exponential interactions". Adjacent particles are connected by strings. Let the positions of the particles at time t be $q_0(t)$, $q_1(t)$, $\ldots$, $q_{N-1}(t)$, where $q_n = q_n(t)$ is the displacement at the moment t of the n-th particle from its equilibrium position. We assume that each particle has mass 1. The momentum of the n-th particle at time t is therefore $p_n = \dot{q}_n$. The Hamiltonian function is defined to be

$$H = \frac{1}{2}\sum_{n=0}^{N-1} p_n^2 + \sum_{n=0}^{N-2} e^{q_n - q_{n+1}}.$$

The Hamiltonian system

$$\dot{q}_n = \frac{\partial H}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial H}{\partial q_n}$$

is

$$\dot{q}_n = p_n, \quad n = 0, 1, \ldots, N-1,$$
$$\dot{p}_0 = -e^{q_0 - q_1},$$
$$\dot{p}_n = e^{q_{n-1} - q_n} - e^{q_n - q_{n+1}}, \quad n = 1, 2, \ldots, N-2,$$
$$\dot{p}_{N-1} = e^{q_{N-2} - q_{N-1}}.$$

Let us set

$$a_n = \frac{1}{2} e^{(q_n - q_{n+1})/2}, \quad n = 0, 1, \ldots, N-2,$$

$$b_n = -\frac{1}{2} p_n, \quad n = 0, 1, \ldots, N-1.$$

Then the above system can be written in the form

$$\dot{a}_n = a_n(b_{n+1} - b_n), \; \dot{b}_n = 2(a_n^2 - a_{n-1}^2), \quad n = 0, 1, \ldots, N-1, \tag{1}$$

with the boundary conditions

$$a_{-1} = a_{N-1} = 0. \tag{2}$$

If we define the $N \times N$ matrices J and A by

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \tag{3}$$

$$A = \begin{bmatrix} 0 & -a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & 0 & -a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & 0 & -a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & 0 \end{bmatrix}, \tag{4}$$

then the system (1) with the boundary conditions (2) is equivalent to the Lax equation

$$\frac{d}{dt} J = [J, A] = JA - AJ. \tag{5}$$

The system (1), (2) is considered subject to the initial conditions

$$a_n(0) = a_n^0, \; b_n(0) = b_n^0, \quad n = 0, 1, \ldots, N-1, \tag{6}$$

where a_n^0, b_n^0 are given complex numbers such that $a_n^0 \neq 0$ $(n = 0, 1, \ldots, N-2)$, $a_{N-1}^0 = 0$.

Existence of the Lax representation (5) allows to solve the nonlinear initial-boundary value problem (1), (2), (6) by means of the inverse spectral problem method.

2. SPECTRAL DATA FOR COMPLEX JACOBI MATRICES AND INVERSE SPECTRAL PROBLEM

An $N \times N$ complex Jacobi matrix is a matrix of the form

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-3} & a_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} \end{bmatrix}, \qquad (7)$$

where for each n, a_n and b_n are arbitrary complex numbers such that a_n is different from zero:

$$a_n,\ b_n \in \mathbb{C},\ a_n \neq 0. \qquad (8)$$

If the entries of the matrix J are real, then the eigenvalues of J are real and simple. However, if the matrix J is complex, then its eigenvalues may be nonreal and multiple. Let $R(\lambda) = (J - \lambda I)^{-1}$ be the resolvent of the matrix J (by I we denote the identity matrix of needed dimension) and e_0 be the N-dimensional vector with the components $1, 0, \dots, 0$. The rational function

$$w(\lambda) = -\left((R(\lambda)e_0, e_0\right) = \left((\lambda I - J)^{-1}e_0, e_0\right) \qquad (9)$$

we call the *resolvent function* of the matrix J.

Denote by $\lambda_1, \dots, \lambda_p$ all the distinct eigenvalues of the matrix J and by $m_1, \dots, m_p$ their multiplicities, respectively, as the roots of the characteristic polynomial $\det(J - \lambda I)$, so that $1 \leq p \leq N$ and $m_1 + \ldots + m_p = N$. We can decompose the rational function $w(\lambda)$ into partial fractions to get

$$w(\lambda) = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \frac{\beta_{kj}}{(\lambda - \lambda_k)^j}, \qquad (10)$$

where β_{kj} are some complex numbers uniquely determined by the matrix J.

Definition 1. The collection of the quantities

$$\{\lambda_k,\ \beta_{kj}\ (j = 1, \dots, m_k,\ k = 1, \dots, p)\}, \qquad (11)$$

we call the spectral data of the matrix J. For each $k \in \{1, \dots, p\}$ the (finite) sequence

$$\{\beta_{k1}, \dots, \beta_{km_k}\}$$

we call the normalizing chain (of the matrix J) associated with the eigenvalue λ_k.

Let us indicate a convenient way for computation of the spectral data of complex Jacobi matrices. For this purpose we should describe the resolvent function $w(\lambda)$ of the Jacobi matrix. Given a Jacobi matrix J of the form (7) with the entries (8), consider the eigenvalue problem $Jy = \lambda y$ for a column vector $y = \{y_n\}_{n=0}^{N-1}$, that is equivalent to the second order linear difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \qquad (12)$$
$$n \in \{0, 1, \dots, N-1\}, \quad a_{-1} = a_{N-1} = 1,$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0.$$

Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of equation (12) satisfying the initial conditions

$$P_{-1}(\lambda) = 0, \quad P_0(\lambda) = 1; \qquad (13)$$

$$Q_{-1}(\lambda) = -1, \quad Q_0(\lambda) = 0. \qquad (14)$$

For each $n \geq 0$, $P_n(\lambda)$ is a polynomial of degree n and is called a polynomial of first kind and $Q_n(\lambda)$ is a polynomial of degree $n-1$ and is known as a polynomial of second kind. The equality

$$\det(J - \lambda I) = (-1)^N a_0 a_1 \cdots a_{N-2} P_N(\lambda) \qquad (15)$$

holds so that the eigenvalues of the matrix J coincide with the zeros of the polynomial $P_N(\lambda)$.

In [16] it is shown that the entries $R_{nm}(\lambda)$ of the matrix $R(\lambda) = (J - \lambda I)^{-1}$ (resolvent of J) are of the form

$$R_{nm}(\lambda) = \begin{cases} P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & n \leq m, \\ P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & m \leq n, \end{cases}$$

where $n, m \in \{0, 1, \dots, N-1\}$ and

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}.$$

Therefore according to (9) and using initial conditions (13), (14), we get

$$w(\lambda) = -R_{00}(\lambda) = -M(\lambda) = \frac{Q_N(\lambda)}{P_N(\lambda)}. \qquad (16)$$

Denote by $\lambda_1, \dots, \lambda_p$ all the distinct roots of the polynomial $P_N(\lambda)$ (which coincide by (15) with the eigenvalues of the matrix J) and by $m_1, \dots, m_p$ their multiplicities, respectively:

$$P_N(\lambda) = c(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_p)^{m_p},$$

where c is a constant. We have $1 \leq p \leq N$ and $m_1 + \ldots + m_p = N$. Therefore by (16) decomposition (10) can be obtained by rewriting the rational function $Q_N(\lambda)/P_N(\lambda)$ as the sum of partial fractions.

We can also get another convenient representation for the resolvent function as follows. If we delete the first row and the first column of the matrix J given in (7), then we get the new matrix

$$J^{(1)} = \begin{bmatrix} b_0^{(1)} & a_0^{(1)} & 0 & \cdots & 0 & 0 & 0 \\ a_0^{(1)} & b_1^{(1)} & a_1^{(1)} & \cdots & 0 & 0 & 0 \\ 0 & a_1^{(1)} & b_2^{(1)} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-4}^{(1)} & a_{N-4}^{(1)} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-4}^{(1)} & b_{N-3}^{(1)} & a_{N-3}^{(1)} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-3}^{(1)} & b_{N-2}^{(1)} \end{bmatrix},$$

where

$$a_n^{(1)} = a_{n+1}, \quad n \in \{0, 1, \dots, N-3\},$$
$$b_n^{(1)} = b_{n+1}, \quad n \in \{0, 1, \dots, N-2\}.$$

The matrix $J^{(1)}$ is called the *first truncated matrix* (with respect to the matrix J).

Theorem 2. The equality

$$w(\lambda) = -\frac{\det(J^{(1)} - \lambda I)}{\det(J - \lambda I)}$$

holds.

Proof. Let us denote the polynomials of the first and the second kinds, corresponding to the matrix $J^{(1)}$, by $P_n^{(1)}(\lambda)$ and $Q_n^{(1)}(\lambda)$, respectively. It is easily seen that

$$P_n^{(1)}(\lambda) = a_0 Q_{n+1}(\lambda), \qquad (17)$$

$$Q_n^{(1)}(\lambda) = \frac{1}{a_0}\{(\lambda - b_0)Q_{n+1}(\lambda) - P_{n+1}(\lambda)\},$$
$$n \in \{0, 1, \ldots, N-1\}.$$

Indeed, both sides of each of these equalities are solutions of the same difference equation

$$a_{n-1}^{(1)} y_{n-1} + b_n^{(1)} y_n + a_n^{(1)} y_{n+1} = \lambda y_n,$$
$$n \in \{0, 1, \ldots, N-2\},\ a_{N-2}^{(1)} = 1,$$

and the sides coincide for $n = -1$ and $n = 0$. Therefore the equalities hold by the uniqueness theorem for solutions.

Consequently, taking into account (15) for the matrix $J^{(1)}$ instead of J and using (17), we find

$$\begin{aligned}\det(J^{(1)} - \lambda I)\\ &= (-1)^{N-1} a_0^{(1)} a_1^{(1)} \cdots a_{N-3}^{(1)} P_{N-1}^{(1)}(\lambda)\\ &= (-1)^{N-1} a_1 \cdots a_{N-2} a_0 Q_N(\lambda).\end{aligned}$$

Comparing this with (15), we get

$$\frac{Q_N(\lambda)}{P_N(\lambda)} = -\frac{\det(J^{(1)} - \lambda I)}{\det(J - \lambda I)}$$

so that the statement of the theorem follows by (16).

The inverse problem is stated as follows:

(i) To see if it is possible to reconstruct the matrix J, given its spectral data (11). If it is possible, to describe the reconstruction procedure.

(ii) To find the necessary and sufficient conditions for a given collection (11) to be spectral data for some matrix J of the form (7) with entries belonging to the class (8).

This problem was solved recently by the author in [16] and the following results were established.

Given the collection (11) define the numbers

$$s_l = \sum_{k=1}^{p} \sum_{j=1}^{m_k} \binom{l}{j-1} \beta_{kj} \lambda_k^{l-j+1}, \tag{18}$$

where $\binom{l}{j-1}$ is a binomial coefficient and we put $\binom{l}{j-1} = 0$ if $j - 1 > l$. Using the numbers s_l let us introduce the determinants

$$D_n = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}, \quad n = 0, 1, \ldots, N. \tag{19}$$

Theorem 3. Let an arbitrary collection (11) of complex numbers be given, where $\lambda_1, \ldots, \lambda_p$ $(1 \le p \le N)$ are distinct, $1 \le m_k \le N$, and $m_1 + \ldots + m_p = N$. In order for this collection to be the spectral data for some Jacobi matrix J of the form (7) with entries belonging to the class (8), it is necessary and sufficient that the following two conditions be satisfied:

(i) $\sum_{k=1}^{p} \beta_{k1} = 1$;

(ii) $D_n \neq 0$, for $n \in \{1, 2, \ldots, N-1\}$, and $D_N = 0$, where D_n is the determinant defined by (19), (18).

Under the conditions of Theorem 3 the entries a_n and b_n of the matrix J for which the collection (11) is spectral data, are recovered by the formulas

$$a_n = \pm \left(D_{n-1} D_{n+1}\right)^{1/2} D_n^{-1}, \tag{20}$$
$$n \in \{0, 1, \ldots, N-2\},\ D_{-1} = 1,$$

$$b_n = \Delta_n D_n^{-1} - \Delta_{n-1} D_{n-1}^{-1}, \tag{21}$$
$$n \in \{0, 1, \ldots, N-1\},\ \Delta_{-1} = 0,\ \Delta_0 = s_1,$$

where D_n is defined by (19) and (18), and Δ_n is the determinant obtained from the determinant D_n by replacing in D_n the last column by the column with the components $s_{n+1}, s_{n+2}, \ldots, s_{2n+1}$.

Remark 4. It follows from the above solution of the inverse problem that the matrix (7) is not uniquely restored from the spectral data. This is linked with the fact that the a_n are determined from (20) uniquely up to a sign. To ensure that the inverse problem is uniquely solvable, we have to specify additionally a sequence of signs + and −. Namely, let $\{\sigma_1, \sigma_2, \ldots, \sigma_{N-1}\}$ be a given finite sequence, where for each $n \in \{1, 2, \ldots, N-1\}$ the σ_n is + or −. We have 2^{N-1} such different sequences. Now to determine a_n uniquely from (20) for $n \in \{0, 1, \ldots, N-2\}$ we can choose the sign σ_n when extracting the square root. In this way we get precisely 2^{N-1} distinct Jacobi matrices possessing the same spectral data. The inverse problem is solved uniquely from the data consisting of the spectral data and a sequence $\{\sigma_1, \sigma_2, \ldots, \sigma_{N-1}\}$ of signs + and −.

3. THE TIME EVOLUTION OF THE SPECTRAL DATA AND SOLVING PROCEDURE FOR THE TODA LATTICE

Theorem 5. Let $\{a_n(t), b_n(t)\}$ be a solution of (1), (2) and $J = J(t)$ be the Jacobi matrix defined by this solution according to (3). Then there exists an ivertible $N \times N$ matrix-function $X(t)$ such that

$$X^{-1}(t) J(t) X(t) = J(0) \quad \text{for all } t. \tag{22}$$

Proof. Let $A(t)$ be defined according to (4). Then the Lax equation (5) holds. Denote by $X = X(t)$ the matrix solution of the initial value problem

$$\dot{X}(t) = -A(t) X(t), \tag{23}$$
$$X(0) = I. \tag{24}$$

Such solution $X(t)$ exists and is unique. From the Liouville formula

$$\det X(t) = \exp\left(-\int_0^t trA(\tau) d\tau\right)$$

it follows that $\det X(t) \neq 0$ so that the matrix $X(t)$ is invertible. Using the formula

$$\frac{dX^{-1}}{dt} = -X^{-1} \frac{dX}{dt} X^{-1},$$

equation (23), and the Lax equation (5), we have

$$\begin{aligned}\frac{d}{dt}\left(X^{-1} J X\right) &= \frac{dX^{-1}}{dt} J X + X^{-1} \frac{dJ}{dt} X + X^{-1} J \frac{dX}{dt}\\ &= X^{-1} A X X^{-1} J X + X^{-1} \frac{dJ}{dt} X - X^{-1} J A X\\ &= X^{-1} \left(\frac{dJ}{dt} - [JA - AJ]\right) X = 0.\end{aligned}$$

Therefore $X^{-1}(t) J(t) X(t)$ does not depend on t and by initial condition (24) we get (22). The proof is complete.

It follows from (22) that for any t the characteristic polynomials of the matrices $J(t)$ and $J(0)$ coincide. Therefore we arrive at the following statement.

Corollary 6. The eigenvalues of the matrix $J(t)$, as well as their multiplicities, do not depend on t.

Remark 7. From the skew-symmetry $A^T = -A$ of the matrix $A(t)$ defined by (4) it follows that for the solution $X(t)$ of (23), (24) we have $X^{-1}(t) = X^T(t)$, where T stands for the matrix transpose. Indeed, by equation (23),

$$\frac{d}{dt}\left(X^T X\right) = \frac{dX^T}{dt}X + X^T\frac{dX}{dt}$$
$$= -X^T A^T X - X^T AX = X^T AX - X^T AX = 0.$$

Therefore $X^T(t)X(t)$ does not depend on t and by initial condition (24) we get $X^T(t)X(t) = X^T(0)X(0) = I$.

Theorem 8. The dynamics of the solution of the Toda equations (1), (2) with the arbitrary complex initial data (6) corresponds to the following evolution of the resolvent function $w(\lambda; t)$ of the matrix $J(t)$:

$$w(\lambda;t) = \frac{e^{2\lambda t}}{S(t)}w(\lambda;0) - \frac{2}{S(t)}\int_0^t S(u)e^{2\lambda(t-u)}du, \tag{25}$$

where

$$S(t) = \frac{1}{2\pi i}\int_\Gamma w(z;0)e^{2zt}dz \tag{26}$$

in which Γ is any closed contour that encloses all the eigenvalues of the matrix $J(0)$.

By Corollary 6 the eigenvalues of the matrix $J(t)$ and their multiplicities do not depend on t. However, the normalazing numbers of the matrix $J(t)$ will depend on t. The following theorem describes this dependence.

Theorem 9. For the normalizing numbers

$$\beta_{kj}(t) \quad (j = 1, \ldots, m_k;\ k = 1, \ldots, p)$$

of the matrix $J(t)$ the following evolution holds:

$$\beta_{kj}(t) = \frac{e^{2\lambda_k t}}{S(t)}\sum_{s=j}^{m_k}\beta_{ks}(0)\frac{(2t)^{s-j}}{(s-j)!}, \tag{27}$$

where

$$S(t) = \sum_{k=1}^{p} e^{2\lambda_k t}\sum_{j=1}^{m_k}\beta_{kj}(0)\frac{(2t)^{j-1}}{(j-1)!}. \tag{28}$$

Proof. According to (10), we have

$$w(\lambda;t) = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}(t)}{(\lambda-\lambda_k)^j}. \tag{29}$$

We will deduce (27), (28) from (25), (26). First we compute the function $S(t)$ in (26). We use the known formula

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta,$$

where γ is the boundary of the disk $D = \{\zeta \in \mathbb{C} : |\zeta - z| \leq \delta\}$ with $\delta > 0$ and $f(z)$ is analytic in a domain of $\mathbb{C}$ containing D.

For any $k \in \{1, \ldots, p\}$ we set $\Gamma_k = \{z \in \mathbb{C} : |z - \lambda_k| = \delta\}$ and choose $\delta > 0$ so small that $\Gamma_1, \ldots, \Gamma_p$ are disjoint and all inside Γ. Then we have, by (26) and (29),

$$S(t) = \frac{1}{2\pi i}\sum_{k=1}^{p}\sum_{j=1}^{m_k}\beta_{kj}(0)\int_\Gamma\frac{e^{2zt}}{(z-\lambda_k)^j}dz.$$

Further,

$$\frac{1}{2\pi i}\int_\Gamma\frac{e^{2zt}}{(z-\lambda_k)^j}dz = \frac{1}{2\pi i}\int_{\Gamma_k}\frac{e^{2zt}}{(z-\lambda_k)^j}dz$$
$$= \frac{(2t)^{j-1}}{(j-1)!}e^{2\lambda_k t}.$$

Therefore for $S(t)$ we have the formula (28).

Now we rewrite (25) in the form, using (29),

$$\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}(t)}{(\lambda-\lambda_k)^j} = \frac{e^{2\lambda t}}{S(t)}\sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}(0)}{(\lambda-\lambda_k)^j} - \frac{2}{S(t)}\int_0^t S(u)e^{2\lambda(t-u)}du. \tag{30}$$

Take any $l \in \{1, \ldots, p\}$ and $s \in \{1, \ldots, m_l\}$, multiply (30) by $(2\pi i)^{-1}(\lambda - \lambda_l)^{s-1}$ and integrate both sides over Γ_l to get

$$\frac{1}{2\pi i}\sum_{j=1}^{m_l}\beta_{lj}(t)\int_{\Gamma_l}\frac{d\lambda}{(\lambda-\lambda_l)^{j-s+1}}$$
$$= \frac{1}{2\pi i}\frac{1}{S(t)}\sum_{j=1}^{m_l}\beta_{lj}(0)\int_{\Gamma_l}\frac{e^{2\lambda t}d\lambda}{(\lambda-\lambda_l)^{j-s+1}}.$$

We have used the fact that the integral of the second term of the right side of (30) gave zero because of the analyticity of that term with respect to λ. Hence

$$\beta_{ls}(t) = \frac{e^{2\lambda_l t}}{S(t)}\sum_{j=s}^{m_l}\beta_{lj}(0)\frac{(2t)^{j-s}}{(j-s)!} \tag{31}$$

because

$$\frac{1}{2\pi i}\int_{\Gamma_l}\frac{d\lambda}{(\lambda-\lambda_l)^{j-s+1}} = \begin{cases} 1 \text{ if } j = s, \\ 0 \text{ if } j \neq s, \end{cases}$$

$$\frac{1}{2\pi i}\int_{\Gamma_l}\frac{e^{2\lambda t}d\lambda}{(\lambda-\lambda_l)^{j-s+1}} = \begin{cases} 0 & \text{if } j < s, \\ \dfrac{(2t)^{j-s}}{(j-s)!}e^{2\lambda_l t} & \text{if } j \geq s. \end{cases}$$

Now, (31) means that (27) holds.

We get the following procedure for solving the problem (1), (2), (6). From the initial data $\{a_n^0, b_n^0\}$ we form the Jacobi matrix $J(0)$ and compute the spectral data

$$\{\lambda_k,\ \beta_{kj}(0)\ (j = 1, \ldots, m_k, k = 1, \ldots, p)\} \tag{32}$$

of the matrix $J(0)$. Then we make up the time-dependent spectral data

$$\{\lambda_k,\ \beta_{kj}(t)\ (j = 1, \ldots, m_k, k = 1, \ldots, p)\}, \tag{33}$$

where $\beta_{kj}(t)$ is obtained from (32) by (27), (28). Finally, solving the inverse spectral problem with respect to (33) we construct a Jacobi matrix $J(t)$. The entries $\{a_n(t), b_n(t)\}$ of the matrix $J(t)$ give a solution of problem (1), (2), (6).

4. EXAMPLES

1. In the case $N = 2$, equations (1), (2) become

$$\dot{a}_0 = a_0(b_1 - b_0), \quad \dot{b}_0 = 2(a_0^2 - a_{-1}^2),$$
$$\dot{a}_1 = a_1(b_2 - b_1), \quad \dot{b}_1 = 2(a_1^2 - a_0^2),$$
$$a_{-1} = a_1 = 0.$$

This system in turn is equivalent to

$$\dot{a}_0 = a_0(b_1 - b_0), \quad \dot{b}_0 = 2a_0^2, \quad \dot{b}_1 = -2a_0^2. \tag{34}$$

with respect to a_0, b_0, and b_1. Let us take initial conditions in the form

$$a_0(0) = ic, \quad b_0(0) = \lambda_0 + c, \quad b_1(0) = \lambda_0 - c,$$

where λ_0 and c are arbitrary complex numbers and $c \neq 0$. We have

$$J(0) = \begin{bmatrix} \lambda_0 + c & ic \\ ic & \lambda_0 - c \end{bmatrix}.$$

Hence

$$\det(J(0) - \lambda I) = (\lambda - \lambda_0)^2,$$

$$w(\lambda; 0) = -\frac{\det(J^{(1)}(0) - \lambda I)}{\det(J(0) - \lambda I)} = -\frac{\lambda_0 - c - \lambda}{(\lambda - \lambda_0)^2}$$

$$= \frac{1}{\lambda - \lambda_0} + \frac{c}{(\lambda - \lambda_0)^2}.$$

Therefore λ_0 is a double eigenvalue of $J(0)$ and the spectral data of $J(0)$ is the collection

$$\{\lambda_0,\ \beta_{11}(0) = 1,\ \beta_{12}(0) = c\}.$$

By (27), (28) we find

$$\beta_{11}(t) = 1, \quad \beta_{12}(t) = \frac{c}{1 + 2ct}.$$

Solving the inverse spectral problem we get

$$J(t) = \begin{bmatrix} \lambda_0 + \dfrac{c}{1+2ct} & \dfrac{ic}{1+2ct} \\ \dfrac{ic}{1+2ct} & \lambda_0 - \dfrac{c}{1+2ct} \end{bmatrix}.$$

Hence the desired solution is

$$a_0(t) = \frac{ic}{1+2ct}, \quad b_0(t) = \lambda_0 + \frac{c}{1+2ct},$$

$$b_1(t) = \lambda_0 - \frac{c}{1+2ct}.$$

2. Now we take for system (34) initial conditions in the form

$$a_0(0) = \sqrt{c(1-c)}, \quad b_0(0) = 1 - c, \quad b_1(0) = c,$$

where c is an arbitrary complex number such that $c \neq 0$ and $c \neq 1$. We have

$$J(0) = \begin{bmatrix} 1 - c & \sqrt{c(1-c)} \\ \sqrt{c(1-c)} & c \end{bmatrix}.$$

Hence

$$\det(J(0) - \lambda I) = \lambda(\lambda - 1),$$

$$w(\lambda; 0) = -\frac{\det(J^{(1)}(0) - \lambda I)}{\det(J(0) - \lambda I)} = -\frac{c - \lambda}{\lambda(\lambda - 1)}$$

$$= \frac{c}{\lambda} + \frac{1 - c}{\lambda - 1}.$$

Therefore $\lambda = 0$ and $\lambda = 1$ are the eigenvalues of $J(0)$ and the spectral data of $J(0)$ is the collection

$$\{\lambda_1 = 0,\ \lambda_2 = 1,\ \beta_{11}(0) = c,\ \beta_{21}(0) = 1 - c\}.$$

By (27), (28) we find

$$\beta_{11}(t) = \frac{c}{c + (1-c)e^{2t}}, \quad \beta_{21}(t) = \frac{(1-c)e^{2t}}{c + (1-c)e^{2t}}$$

Solving the inverse spectral problem we get that the desired solution is

$$a_0(t) = \frac{e^t\sqrt{c(1-c)}}{c + (1-c)e^{2t}}, \quad b_0(t) = \frac{(1-c)e^{2t}}{c + (1-c)e^{2t}},$$

$$b_1(t) = \frac{c}{c + (1-c)e^{2t}}.$$

REFERENCES

[1] M. Toda, Waves in nonlinear lattice, *Progr. Theoret. Phys Suppl.* 45 (1970) 174–200.

[2] M. Henon, Integrals of the Toda lattice, *Phys. Rev. B* 9 (1974) 1921–1923.

[3] H. Flaschka, The Toda lattice, I, *Phys. Rev. B* 9 (1974) 1924–1925.

[4] Flaschka H. The Toda lattice, II, *Progr. Theoret. Phys.* 1974;51:703–716.

[5] S. V. Manakov, Complete integrability and stochastization in discrete dynamic media, *Zh. Exper. Teor. Fiz.* 67 (1974) 543–555. English translation in *Soviet Physics-JETP* 40 (1975) 269–274.

[6] M. Kac and P. van Moerbeke, On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices, *Adv. Math.* 16 (1975) 160–169.

[7] J. Moser, Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. Math.* 16 (1975) 197–220.

[8] M. Toda, *Theory of nonlinear lattices*, Springer, New York, 1981.

[9] Yu. M. Berezanskii, The integration of semi-infinite Toda chain by means of inverse spectral problem, *Rep. Math. Phys.* 24 (1986) 21–47.

[10] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Math surveys monographs, vol. 72. Amer Math Soc Providence (RI); 2000.

[11] A. I. Aptekarev and A. Branquinbo, Padé approximants and complex high order Toda lattices, *J. Comput. Appl. Math.* 155 (2003) 231–237.

[12] Yu. M. Berezanskii and A. A. Mokhonko, Integration of some differential-difference nonlinear equations using the spectral theory of normal block Jacobi matrices, *Functional Anal. Appl.* 42 (2008) 1–18.

[13] A. Huseynov, On solutions of a nonlinear boundary value problem for difference equations, *Transactions Natl. Acad. Sci. Azerb. Ser. Phys-Tech. Math. Sci.* 28(4) (2008) 43–50.

[14] H. Bereketoglu and A. Huseynov, Boundary value problems for nonlinear second-order difference equations with impulse, *Appl. Math. Mech. (English Ed.)* 30 (2009) 1045–1054.

[15] A. Huseynov, Positive solutions for nonlinear second-order differernce equations with impulse, *Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis* 17 (2010) 115–132.

[16] G. Sh. Guseinov, Inverse spectral problems for tridiagonal N by N complex Hamiltonians, *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)*, 5 (2009), Paper 018, 28 pages.