Numerical Tests In Histopolation By Rational Spline

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Abstract: In this paper with numerical tests the convergence of Newton's method with ordinary method for solving of nonlinear system result of Histopolation problem is compared. Aitken's transform in the ordinary method is used to accelerate ordinary method. An example of HISTOPOLATION application in physics will be presented.

Keywords: Histopolation, nonlinear system, Newton's method, Ordinary method, rational spline.

1. INTRODUCTION

In [1] for each subinterval $[x_{i-1}, x_i]$ of the grid $a = x_0 < x_1 < \cdots < x_n = b$ and given real number z_i that $z_1 < z_2 < \cdots < z_n$ *or* $z_1 > z_2 > \cdots > z_n$ finding rational spline of the form $s(x) = \dfrac{a_i + b_i(x - x_{i-1})}{1 + d_i(x - x_{i-1})}$ with $1 + d_i(x - x_{i-1}) > 0, x \in [x_{i-1}, x_i]$ and satisfying the area-matching conditions (that is known Histopolation condition) $\int_{x_{i-1}}^{x_i} s(x)dx = z_i(x_i - x_{i-1})$ and boundary conditions leads to a nonlinear system. This system done successfully by the ordinary iteration method or Newton's method and since convergence of ordinary method in comparison with Newton's method is very slow, Aitken's transform is used. In [1] is proved Histopolation by rational spline is also an interpolation by rational spline and in [2] Carnicer and Dahmen showed that any c^1-smooth strict convexity preserving interpolation method can not be linear but rational spline for a strictly monotone function f is also strictly monotone. in [3] one field of Histopolation by cubic spline in the motion of material point and statistic is stated that in this paper an example of Histopolation by rational spline in physics will be presented.

2. REPRESENTATION OF THE HISTOPOLATION

To find $s \in c^1([a,b])$ in the form $s(x) = \dfrac{a_i + b_i(x - x_{i-1})}{1 + d_i(x - x_{i-1})}, x \in [x_{i-1}, x_i]$, $1 + d_i(x - x_{i-1}) > 0$ that $\int_{x_{i-1}}^{x_i} s(x)dx = z_i(x_i - x_{i-1})$ for $i = 1, \cdots, n$ which involves n conditions for given $z_i, i = 1,..., n$ by setting

boundary conditions $s'(x_0) = \alpha$ and $s'(x_n) = \beta$ we have 3n conditions (since $s \in c^1([a,b])$ which involves 2(n-1) conditions) and 3n parameters.

By setting $s'(x_i) = m_i, i = 0,..., n$ and $h_i = x_i - x_{i-1}$ $i = 1, \cdots, n$ and $c_i = b_i - a_i d_i$ have $c_i = m_{i-1}$ and $d_i = \dfrac{1}{h_i}((\dfrac{m_{i-1}}{m_i})^{\frac{1}{2}} - 1)$ for $m_i \neq 0$. Now with Histopolation condition, s(x) in the case $d_i \neq 0$ is expressed with $x = x_{i-1} + th_i$ the representation

$$s(x) = z_i + h_i \frac{m_{i-1}}{((\frac{m_{i-1}}{m_i})^{\frac{1}{2}} - 1)^2} \log(\frac{m_{i-1}}{m_i})^{\frac{1}{2}} -$$

$$h_i \frac{m_{i-1}}{((\frac{m_{i-1}}{m_i})^{\frac{1}{2}} - 1)(1 + t((\frac{m_{i-1}}{m_i})^{\frac{1}{2}} - 1))}.$$

And in the case $d_i = 0$ the representation

$$s(x) = z_i + m_i(x - (x_{i-1} + \frac{h_i}{2})).$$

Now express the continuity of s gives:

$$m_i(h_i\varphi((\frac{\mathrm{m}_{\mathrm{i-1}}}{\mathrm{m}_{\mathrm{i}}})^{\frac{1}{2}})+h_{i+1}\varphi((\frac{\mathrm{m}_{\mathrm{i+1}}}{\mathrm{m}_{\mathrm{i}}})^{\frac{1}{2}})))=\delta_i,$$

Where $\delta_i = z_{i+1} - z_i$ and

$$\varphi(x)=\begin{cases}\dfrac{x^2(\log x-1)+x}{(x-1)^2} & x>0, x\neq 1\\ \dfrac{1}{2} & x=1\end{cases}$$

This system could be written in the form:

$$\Psi_0(m)\equiv m_0-\alpha=0,$$

$$\Psi_i(m)\equiv m_i(h_i\varphi((\frac{\mathrm{m}_{\mathrm{i-1}}}{\mathrm{m}_{\mathrm{i}}})^{\frac{1}{2}})+h_{i+1}\varphi((\frac{\mathrm{m}_{\mathrm{i+1}}}{\mathrm{m}_{\mathrm{i}}})^{\frac{1}{2}}))-\delta_i=0,$$

$$\Psi_n(m)\equiv m_n-\beta=0.$$

This system can assumes the form $\Psi(m)=(\Psi_0(m),...,\Psi_n(m))=0$.

In the Newton's method for solving of this nonlinear system $m^{k+1}=m^k-\Psi'^{-1}(m^k)\Psi(m^k)$

And initial values $m_i^{\,0}=\dfrac{2(z_{i+1}-z_i)}{h_i+h_{i+1}}, i=1,...,n-1$. The entries of the matrix Ψ' are:

$$\begin{cases}\dfrac{d\Psi_i}{dm_{i-1}}=\dfrac{1}{2}h_i\dfrac{1}{u_i}\varphi'(u_i)\\ \dfrac{d\Psi_i}{dm_i}=h_i\varphi'(u_i)-\dfrac{1}{2}h_iu_i\varphi'(u_i)\\ \qquad +h_{i+1}\varphi(v_i)-\dfrac{1}{2}h_{i+1}v_i\varphi'(v_i)\\ \dfrac{d\Psi_i}{dm_{i+1}}=\dfrac{1}{2}h_{i+1}\dfrac{1}{v_i}\varphi'(v_i).\end{cases}$$

Proposition 2.1: The diagonal of the matrix in Newton's method is dominant (and so is non-singular) if $\dfrac{m_{i-1}}{m_i}\geq 0.54, \dfrac{m_{i+1}}{m_i}\geq 0.54$ and at least one of these inequalities is strict.

Theorem 2.2: (ordinary method) the equation $m=\Psi(m)$

Where $\Psi(m)=m-\alpha\Phi(m)$ for some $K>0$, $\|\Phi'(m)X\|\leq Kh^2X, \forall X\in R^{n+1}$ then the condition $0<\alpha<\dfrac{2\gamma}{K^2h^2}$ for some $\gamma>0$ implies $\|\Psi'(m)\|<1$ which it follows that Ψ is a contraction.

Theorem 2.3: (Aitken's Theorem) suppose there exists k, $|k|<1$ such that for the sequence $\{x_i\}, x_i\neq\varepsilon$, $x_{i+1}-\varepsilon=(k+\delta_i)(x_i-\varepsilon)$, $\lim_{i\to\infty}\delta_I=0,$ holds, then $x_i'=x_i-\dfrac{(x_{i+1}-x_i)^2}{x_{i+1}-2x_{i+1}+x_i}$ all exist for sufficiently large i, and $\lim_{i\to\infty}\dfrac{x_i'-\varepsilon}{x_i-\varepsilon}=0$ i.e. sequence $\{x_i'\}$ in general converges faster towards ε than the original sequence of value x_i.

The proof of the Theorems and proposition is presented in [1,4].

3. NUMERICAL TESTS

We Histopolated $f(x)=5\sin(x)$ on $x\in[0,\pi/2]$ (that $h_i=\dfrac{b-a}{m}$, $a=0, b=\pi/2, m=17, x_i=a+ih$ and $z_i=\dfrac{1}{h_i}\int_{x_{i-1}}^{x_i}f(x)dx$) by Newton's method with starting values $m_i^{\,0}=\dfrac{2(z_i-z_{i-1})}{h_i+h_{i-1}}$ and by ordinary method $m^{k+1}=m^k-\alpha.\Psi(m^k)$ that ordinary method is very slow (as see in table) so we use of Aitken's method $m_i^{\,k}-\dfrac{(m_i^{\,k+1}-m_i^{\,k})^2}{m_i^{\,k+2}-2m_i^{\,k+1}-m_i^{\,k}}$ replace of $m_i^{\,k}$. Results are present in some tables:

Table 1. Ordinary method $(\alpha=1)$

Iteration, k,	$\|m^{k+1}-m^k\|_\infty$	$\|\Psi(m)\|_\infty$
5	0.03216164038352	0.00405471440103
10	0.00701849392013	0.00288127377185
50	0.00022126816921e-2	0.58353014887749e-3
100	0.00001355593527e-4	0.81559619699384e-4
150	0.00002142317682e-5	0.13252326695606e-4
200	0.00001451523254e-6	0.03488457766682e-5
300	0.00002345121984e-8	0.90294774601762e-7

Table 2. Ordinary method $(\alpha = .01)$

Iteration k	$\|\Psi(m)\|_\infty$
5	0.00530676893421
10	0.00528947111786
50	0.00515302380484
100	0.00498721971706

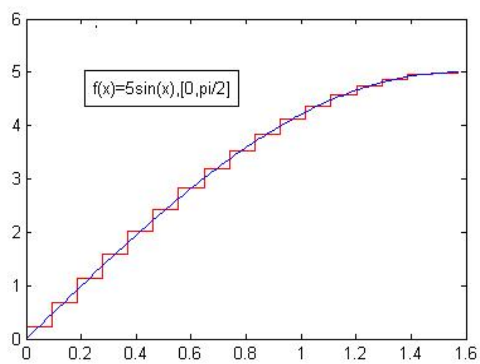
Table 3. Newton's method

Iteration, k	$\|\Psi(m)\|_\infty$
1	0.00098665512354
2	0.00035743289911e-3
5	0.00003417266470e-6
10	0.00055178084324e-10

Table 4. Aitken's transform in ordinary method

Iteration k	$\|\Psi(m)\|_\infty$
5	0.00096250305270
10	0.00094856233684
50	0.00015739188429
100	0.02251962318738e-3
150	0.03691502476685e-4

Figure 1. Histopolation of $f(x) = 5\sin(x)$



4. APPLICATION OF HISTOPOLATION

Histopolation can be utilized for formulation of motion of a particle. The problem is how to find S(t) as function of time t assuming that at some times $t = t_i, (i = 0,\ldots,n)$ be in the positions $g(t_i) = g_i$.

The velocity of a particle is written as:

$$s(t) = \frac{dx}{dt} \rightarrow \int_{t_{i-1}}^{t_i} s(t)dt = g_i - g_{i-1} \qquad i = 1,\ldots,n$$

Where by introducing

$$z_i = \frac{g_i - g_{i-1}}{h_i}$$

The problem is transformed to the Histopolation one.

Example: consider a particle moving in a static electrical field. Due to electrostatic force, the particle gets an acceleration that at some times $t = t_i$ we have the following table for g_i:

Table 5.

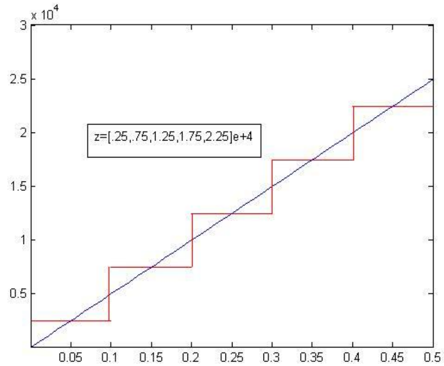
t_i	g_i
0	0
0.1	0.25e+3
0.2	1e+3
0.3	2.25e+3
0.4	4e+3
0.5	6.25e+3

In physics we have $v(t) = \frac{qE}{m}t + v_0$, that have to determine q, E, m. but in Histopolation for finding $s(t) = v$ we have table 6:

Table 6.

t	z_i
0<t<0.1	0.25e+4
0.1<t<0.2	0.75e+4
0.2<t<0.3	1.25e+4
0.3<t<0.4	1.75e+4
0.4<t<0.5	2.25e+4

Figure 2. Histopolation of table 6



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