C-Integrable q-Burgers Type Equation and Exact Solutions

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Abstract:
By using Jackson's q-exponential function we introduce the generating function, the recursive formulas and the second order q-differential equation for the q-Hermite polynomials. This allows us to solve the q-heat equation in terms of q-Kampe de Feriet polynomials with arbitrary N moving zeroes, and to find operator solution for the Initial Value Problem for the q-heat equation. By the q-analog of the Cole-Hopf transformation we construct the q-Burgers type nonlinear heat equation with quadratic dispersion and the cubic nonlinearity. In $q \to 1$ limit it reduces to the standard Burgers equation. Exact solutions for the q-Burgers equation in the form of moving poles, singular and regular q-shock soliton solutions are obtained. Novel, self-similar property for stationary regular q-shock soliton solution is found.

Keywords: q-calculus, q-Hermite polynomials, q-Burgers equation, q-Heat equation

1. INTRODUCTION

It is well known that the Burgers' equation in one dimension could be linearized by the Cole-Hopf transformation in terms of the linear heat equation. It allows one to solve the initial value problem for the Burgers equation and to get exact solutions in the form of shock solitons and describe their scattering. In the present paper we study the q-differential Burgers type equation with quadratic dispersion and the cubic nonlinearity, and find its linearization in terms of the q-heat equation. In terms of the Jackson's q-exponential function we introduce the q-Hermite and q-Kampe-de Feriet polynomials, representing moving poles solution for the q-Burgers equation. Then we derive the operator solution of the initial value problem for the q-Burgers equation in terms of the IVP for the q-heat equation. We find solutions of our q-Burgers type equation in the form of singular and regular q-shock solitons. It turns out that static q-shock soliton solution shows remarkable self-similarity property in space coordinate x.

2. BASIC Q-CALCULUS

The q-number corresponding to the ordinary number n is defined as, Kac et al. (2002),

$$[n]_q = \frac{q^n - 1}{q - 1}, \tag{1}$$

where q is a parameter, so that n is the limit of $[n]_q$ as $q \to 1$. A few examples of q-numbers are given here: $[0]_q = 0$, $[1]_q = 1$, $[2]_q = 1 + q$, $[3]_q = 1 + q + q^2$. In terms of these q-numbers, the Jackson q-exponential function is defined as

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}. \tag{2}$$

For $q > 1$ it is entire function of x and when $q \to 1$ it reduces to the standard exponential function e^x. The q-exponential function can also be expressed in terms of infinite product

$$e_q(x) = \prod_{n=0}^{\infty} \frac{1}{(1 - (1-q)q^n x)} = \frac{1}{(1 - (1-q)x)_q^{\infty}}, \tag{3}$$

when $q < 1$ and

$$e_q(x) = \prod_{n=0}^{\infty} \left(1 + (1 - \frac{1}{q})q^n x\right) = \left(1 + (1 - \frac{1}{q})x\right)_{1/q}^{\infty}, \tag{4}$$

when $q > 1$. Thus, the q-exponential function for $q < 1$ has infinite set of poles at

$$x_n = \frac{1}{q^n(1-q)}, \quad n = 0, 1, .. \tag{5}$$

and for $q > 1$ the infinite set of zeros at

$$x_n = -\frac{q}{q^n(q-1)}, \quad n = 0, 1, .. \tag{6}$$

The q-derivative is defined as

$$D_q^x f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \tag{7}$$

and when $q \to 1$ it reduces to the standard derivative $D_q^x f(x) \to f'(x)$. Using the definition of the q-derivative one can easily see that

$$D_q^x(ax^n) = a[n]_q x^{n-1}, \tag{8}$$

$$D_q^x e_q(ax) = a e_q(ax). \tag{9}$$

3. Q-HERMITE POLYNOMIALS

We define the q-Hermite polynomials according to the generating function

$$e_q(-t^2)e_q([2]_q tx) = \sum_{n=0}^{\infty} H_n(x;q)\frac{t^n}{[n]_q!}. \tag{10}$$

From this generating function we have the special values

$$H_{2n}(0;q) = (-1)^n \frac{[2n]_q!}{[n]_q!}, \tag{11}$$

$$H_{2n+1}(0;q) = 0, \tag{12}$$

where $[n]_q! = [1]_q[2]_q...[n]_q$, and the parity relation

$$H_n(-x;q) = (-1)^n H_n(x;q). \tag{13}$$

By q-differentiating the generating function (10) according to x and t we have the recurrence relations correspondingly

$$D_x H_n(x;q) = [2]_q[n]_q H_{n-1}(x;q), \tag{14}$$

$$H_{n+1}(x;q) = [2]_q x H_n(x;q) - [n]_q H_{n-1}(qx;q)$$
$$-[n]_q q^{\frac{n+1}{2}} H_{n-1}(\sqrt{q}x;q). \tag{15}$$

Using operator

$$M_q = q^{x\frac{d}{dx}}, \tag{16}$$

so that

$$M_q f(x) = f(qx), \tag{17}$$

relation (15) can be rewritten as

$$H_{n+1}(x;q) = [2]_q x H_n(x;q)$$
$$-[n]_q(M_q + q^{\frac{n+1}{2}} M_{\sqrt{q}})H_{n-1}(x;q). \tag{18}$$

Substituting (14) to (18) we get

$$H_{n+1}(x;q) = \left([2]_q x - \frac{M_q + q^{\frac{n+1}{2}} M_{\sqrt{q}}}{[2]_q} D_x\right) H_n(x;q) \tag{19}$$

By the recursion, starting from $n = 0$ and $H_0(x) = 1$ we have next representation for the q-Hermite polynomials

$$H_n(x;q) = \prod_{k=1}^{n} \left([2]_q x - \frac{M_q + q^{\frac{k}{2}} M_{\sqrt{q}}}{[2]_q} D_x\right) \cdot 1 \tag{20}$$

We notice that the generating function and the form of our q-Hermite polynomials are different from the known ones in the literature, Exton (1983), Cigler et al. (2009), Rajkovic et al. (2001), Negro (1996) . Moreover, our the three term recurrence relation (15) is q-nonlocal and different from the standard local relation for orthogonal polynomial sets, like discrete Al Salam-Carlitz q-polynomials, Ismail (2005).
In the above expression the operator

$$M_q + q^{\frac{n}{2}} M_{\sqrt{q}} = 2q^{\frac{n}{4}} q^{\frac{3}{4}x\frac{d}{dx}} \cosh[(\ln q^{\frac{1}{4}})(x\frac{d}{dx} - n)] \tag{21}$$

is expressible in terms of the q-spherical means as

$$\cosh[(\ln q)x\frac{d}{dx}]f(x) = \frac{1}{2}(f(qx) + f(\frac{1}{q}x)). \tag{22}$$

Using the definition of, Kac et al. (2002),

$$(x-a)_q^n = (x-a)(x-qa)\cdots(x-q^{n-1}a), \ n = 1, 2, ..$$

which now we apply for operators, we should distinguish the direction of multiplication. We consider two cases

$$(x-a)_{q<}^n = (x-a)(x-qa)\cdots(x-q^{n-1}a), \tag{23}$$

and

$$(x-a)_{q>}^n = (x-q^{n-1}a)\cdots(x-qa)(x-a). \tag{24}$$

Then, we can rewrite (20) shortly as

$$H_n(x;q) = \left(([2]_q x - \frac{M_q D_x}{[2]_q}) - q^{\frac{1}{2}}\frac{M_{\sqrt{q}} D_x}{[2]_q}\right)^n_{\sqrt{q}>} \cdot 1$$

First few polynomials are

$$H_0(x;q) = 1,$$

$$H_1(x;q) = [2]_q x,$$

$$H_2(x;q) = [2]_q^2 x^2 - [2]_q,$$

$$H_3(x;q) = [2]_q^3 x^3 - [2]_q^2[3]_q x,$$

$$H_4(x;q) = [2]_q^4 x^4 - [2]_q^2[3]_q[4]_q x^2 + [2]_q[3]_q[2]_{q^2}.$$

When $q \to 1$ these polynomials reduce to the standard Hermite polynomials.

3.1 q-Differential Equation

Applying D_x to both sides of (19) and using recurrence formula (14) we get q-differential equation for q-Hermite polynomials

$$\frac{1}{[2]_q} D_x(M_q + q^{\frac{n+1}{2}} M_{\sqrt{q}})D_x H_n(x;q)$$
$$-[2]_q q x D_x H_n(x;q) + [2]_q[n]_q q H_n(x;q) = 0. \tag{25}$$

3.2 Operator Representation

Proposition 1. We have next identity

$$e_q\left(-\frac{1}{[2]_q^2} D_x^2\right) e_q([2]_q xt) = e_q(-t^2)e_q([2]_q xt). \tag{26}$$

Proof. By q- differentiating the q-exponential function in x

$$D_x^n e_q([2]_q xt) = ([2]_q t)^n e_q([2]_q xt), \tag{27}$$

and combining then to the sum

$$\sum_{n=0}^{\infty} \frac{a^n}{[n]_q!} D_x^{2n} e_q([2]_q xt) = \sum_{n=0}^{\infty} \frac{[2]_q^{2n} a^n t^{2n}}{[n]_q!} e_q([2]_q xt), \tag{28}$$

we have relation

$$e_q(aD_x^2)e_q([2]_q xt) = e_q([2]_q^2 at^2)e_q([2]_q xt). \tag{29}$$

By choosing $a = -1/[2]_q^2$ we get

$$e_q\left(-\frac{1}{[2]_q^2} D_x^2\right) e_q([2]_q xt) = e_q(-t^2)e_q([2]_q xt). \tag{30}$$

Proposition 2. The next identity is valid

$$H_n(x;q) = [2]^n e_q\left(-\frac{1}{[2]_q^2}D_x^2\right)x^n. \qquad (31)$$

Proof. The right hand side of (26) is the generating function for q-Hermite polynomials. Hence expanding both sides in t we get the result.

Proposition 3.

$$e_q\left(-\frac{D_x^2}{[2]_q^2}\right)x^{n+1} =$$

$$\frac{1}{[2]_q}\left([2]_q\,x - \frac{(M_q + q^{\frac{n+1}{2}}M_{\sqrt{q}})D_x}{[2]_q}\right)e_q\left(-\frac{D_x^2}{[2]_q^2}\right)x^n.$$

Proof. we use (31) and relation (19) .

Corollary 4. If function $f(x)$ is analytic and expandable to power series $f(x) = \sum_{n=0}^{\infty}a_n x^n$ then we have next q-Hermite series

$$e_q\left(-\frac{1}{[2]_q^2}D_x^2\right)f(x) = \sum_{n=0}^{\infty}a_n\frac{H_n(x;q)}{[2]_q^n}. \qquad (32)$$

4. Q- KAMPE DE FERIET POLYNOMIALS

Using relation

$$H_n(x,t;q) = (-\nu t)^{\frac{n}{2}}H_n\left(\frac{x}{[2]_q\sqrt{-\nu t}}\right), \qquad (33)$$

and (19) we have recursion formula for q- Kampe-de Feriet polynomials

$$H_{n+1}(x,t;q) = \left(x + (M_q + q^{\frac{n+1}{2}}M_{\sqrt{q}})\nu t D_x\right)H_n(x,t;q).$$

By recursion it gives

$$H_n(x,t;q) = \prod_{k=1}^{n}\left(x + (M_q + q^{\frac{k}{2}}M_{\sqrt{q}})\nu t D_x\right)\cdot 1 \qquad (34)$$

or by notation (24)

$$H_n(x,t;q) = \left((x + M_q\,\nu t\,D_x) + q^{\frac{1}{2}}M_{\sqrt{q}}\,\nu t D_x\right)_{\sqrt{q}>}^{n}\cdot 1$$

First few polynomials are

$$H_0(x,t;q) = 1,$$

$$H_1(x,t;q) = x,$$

$$H_2(x,t;q) = x^2 + [2]_q\,\nu t,$$

$$H_3(x,t;q) = x^3 + [2]_q[3]_q\,\nu t\,x,$$

$$H_4(x,t;q) = x^4 + [3]_q[4]_q\,\nu t\,x^2 + [2]_q[3]_q[2]_{q^2}\nu^2 t^2.$$

5. Q-HEAT EQUATION

We consider the q-heat equation

$$(D_t - \nu D_x^2)\phi(x,t) = 0. \qquad (35)$$

Solution of this equation expanded in terms of parameter k

$$\phi(x,t) = e_q(\nu k^2 t)e_q(kx) = \sum_{n=0}^{\infty}\frac{k^n}{[n]!}H_n(x,t;q), \qquad (36)$$

gives the set of q-Kampe-de Feriet polynomial solutions for the equation. Then we find time evolution of zeroes for these solutions in terms of zeroes $z_k(n,q)$ of q-Hermite polynomials,

$$H_n(z_k(n,q),q) = 0, \qquad (37)$$

so that

$$x_k(t) = [2]z_k(n,q)\sqrt{-\nu t}. \qquad (38)$$

For n=2 we have two zeros determined by q-numbers,

$$z_1(2,q) = \frac{1}{\sqrt{[2]_q}}, \qquad (39)$$

$$z_2(2,q) = \frac{-1}{\sqrt{[2]_q}}, \qquad (40)$$

and moving in opposite directions according to (38). For n=3 we have zeros determined by q-numbers,

$$z_1(3,q) = -\sqrt{\frac{[3]_q}{[2]_q}}, \qquad (41)$$

$$z_2(3,q) = 0, \qquad (42)$$

$$z_1(3,q) = \sqrt{\frac{[3]_q}{[2]_q}}, \qquad (43)$$

two of which are moving in opposite direction according to (38) and one is in the rest.

6. EVOLUTION OPERATOR

Following similar calculations as in Proposition I we have next relation

$$e_q\left(\nu t D_x^2\right)e_q(kx) = e_q(\nu t k^2)e_q(kx). \qquad (44)$$

The right hand side of this expression is the plane wave type solution of the q-heat equation

$$(D_t - \nu D_x^2)\phi(x,t) = 0. \qquad (45)$$

Expanding both sides in power series in k we get q-Kampe de Feriet polynomial solutions of this equation

$$H_n(x,t;q) = e_q\left(\nu t D_x^2\right)x^n. \qquad (46)$$

Consider an arbitrary analytic function $f(x) = \sum_{n=0}^{\infty}a_n x^n$, then function

$$f(x,t) = e_q\left(\nu t D_x^2\right)f(x) = \sum_{n=0}^{\infty}a_n e_q\left(\nu t D_x^2\right)x^n \qquad (47)$$

$$= \sum_{n=0}^{\infty}a_n H_n(x,t;q), \qquad (48)$$

is a time dependent solution of the q-heat equation (45).

According to this we have the evolution operator for the q-heat equation as

$$U(t) = e_q \left(\nu t D_x^2 \right). \tag{49}$$

It allows us to solve the initial value problem

$$(D_t - \nu D_x^2)\phi(x,t) = 0, \tag{50}$$

$$\phi(x, 0^+) = f(x), \tag{51}$$

in the form

$$\phi(x,t) = e_q \left(\nu t D_x^2 \right) \phi(x, 0^+) = e_q \left(\nu t D_x^2 \right) f(x). \tag{52}$$

7. Q-BURGERS' TYPE EQUATION

Let us consider the q-Cole-Hopf transformation

$$u(x,t) = -2\nu \frac{D_x \phi(x,t)}{\phi(x,t)}, \tag{53}$$

then $u(x,t)$ satisfies the q-Burgers' type Equation with quadratic dispersion and cubic nonlinearity

$$D_t u(x,t) - D_x^2 u(x,t) =$$

$$\frac{1}{2} \left[(u(x,qt) - u(x,t)M_q^x) D_x u(x,t) \right] -$$

$$\frac{1}{2} \left[D_x \left(u(qx,t)u(x,t) \right) \right] +$$

$$\frac{1}{4} \left[u(q^2 x, t) - u(x, qt) \right] u(qx,t)u(x,t).$$

When $q \to 1$ it reduces to the standars Burgers' Equation.

7.1 I.V.P. for q-Burgers' Type Equation

Substituting the operator solution (52) to (53) we find operator solution for the q-Burgers type equation in the form

$$u(x,t) = -2\nu \frac{e_q \left(\nu t D_x^2 \right) D_x f(x)}{e_q \left(\nu t D_x^2 \right) f(x)}. \tag{54}$$

This solution corresponds to the initial function

$$u(x, 0^+) = -2\nu \frac{D_x f(x)}{f(x)}. \tag{55}$$

Thus, for arbitrary initial value problem for the q-Burgers equation with $u(x, 0^+) = F(x)$ we need to solve the initial value problem for the q-heat equation with initial function $f(x)$ satisfying the first order q-differential equation

$$(D_x + \frac{1}{2\nu} F(x)) f(x) = 0. \tag{56}$$

8. Q-SHOCK SOLITON SOLUTION

As a solution of q-heat equation we choose first

$$\phi(x,t) = e_q \left(k^2 t \right) e_q \left(kx \right), \tag{57}$$

then we find solution of the q-Burgers equation as

$$u(x,t) = -2\nu k. \tag{58}$$

We notice that for this solution of the q-heat equation, we have infinite set of zeros, and space position of zeros is fixed

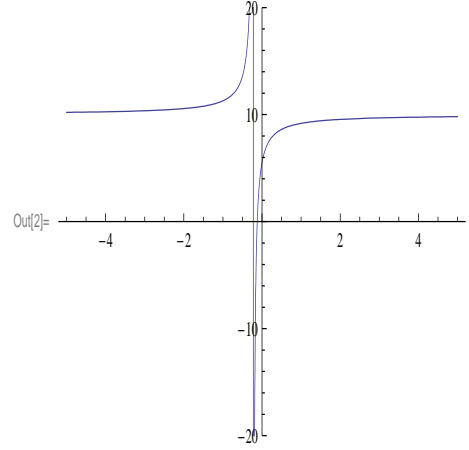

Fig. 1. The singular q-shock soliton for $k_1 = 1$, $k_2 = 10$ and $t = 0$.

during time evolution at points $x_n = -\frac{q^{n+1}}{(q-1)k}, n = 0, 1, ...$ If we choose

$$\phi(x,t) = e_q \left(k_1^2 t \right) e_q \left(k_1 x \right) + e_q \left(k_2^2 t \right) e_q \left(k_2 x \right), \tag{59}$$

then we have the q-Shock soliton solution

$$u(x,t) = -2\nu \frac{k_1 e_q \left(k_1^2 t \right) e_q \left(k_1 x \right) + k_2 e_q \left(k_2^2 t \right) e_q \left(k_2 x \right)}{e_q \left(k_1^2 t \right) e_q \left(k_1 x \right) + e_q \left(k_2^2 t \right) e_q \left(k_2 x \right)}. \tag{60}$$

Due to zeroes of the q-exponential function this expression admits singularities for some values of parameters k_1 and k_2. In Fig.1 we plot the singular q-shock soliton for $k_1 = 1$ and $k_2 = 10$ at time $t = 0$. However for some specific values of the parameters we found the regular q-shock soliton solution. We introduce the q-hyperbolic function

$$\cosh_q(x) = \frac{e_q(x) + e_q(-x)}{2}, \tag{61}$$

or

$$\cosh_q(x) = \frac{1}{2} \left(e_q(x) + \frac{1}{e_{\frac{1}{q}}(x)} \right), \tag{62}$$

then by using infinite product representation for q-exponential function we have

$$\cosh_q(x) = \frac{1}{2} \left(\left(1 + (1 - \frac{1}{q})x \right)_{1/q}^{\infty} + \left(1 - (1 - \frac{1}{q})x \right)_q^{\infty} \right).$$

From (5),(6) we find that zeroes of the first product are located on negative axis x, while for the second product on the positive axis x. Therefore the function has no zeros for real x and $\cosh_q(0) = 1$.

If $k_1 = 1$, and $k_2 = -1$, the time dependent factors in nominator and the denominator of (60) cancel each other and we have the stationary shock soliton

$$u(x,t) = -2\nu \frac{e_q(x) - e_q(-x)}{e_q(x) + e_q(-x)} \equiv -2\nu \tanh_q(x). \tag{63}$$

Due to above consideration this function has no singularity on real axis and we have regular q-shock soliton.

In Fig.2, Fig.3 and Fig.4 we plot the regular q-shock soliton for $k_1 = 1$ and $k_2 = -1$ at different ranges of
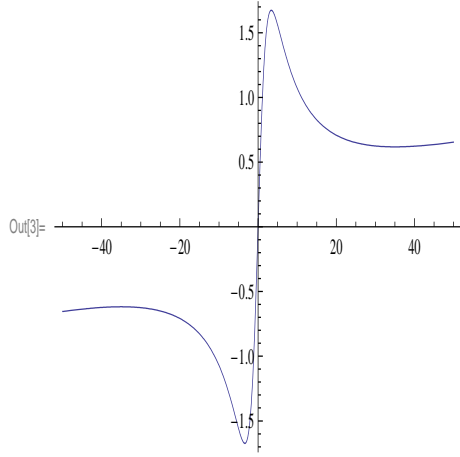
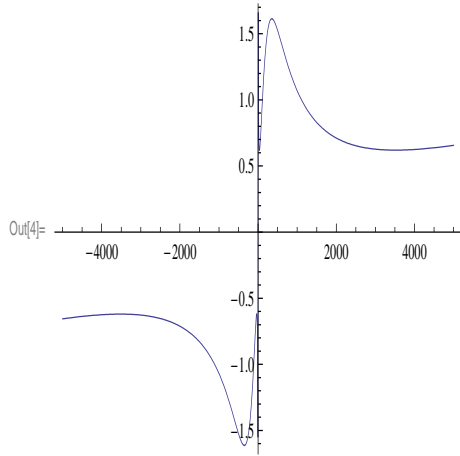Fig. 2. The regular q-shock soliton for $k_1 = 1$, $k_2 = -1$, at range (-50, 50)

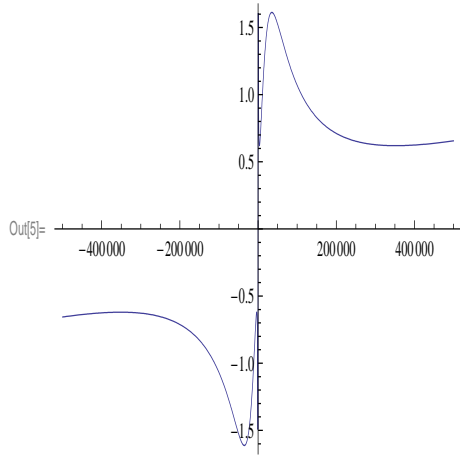Fig. 3. The regular q-shock soliton for $k_1 = 1$, $k_2 = -1$ at range (-5000, 5000)

Fig. 4. The regular q-shock soliton for $k_1 = 1$, $k_2 = -1$ at range (-500000, 500000)

x. It is remarkable fact that the structure of our shock soliton shows self-similarity property in space coordinate x. Indeed at the ranges of parameter $x = 50, 5000, 500000$ the structure of shock looks almost the same.
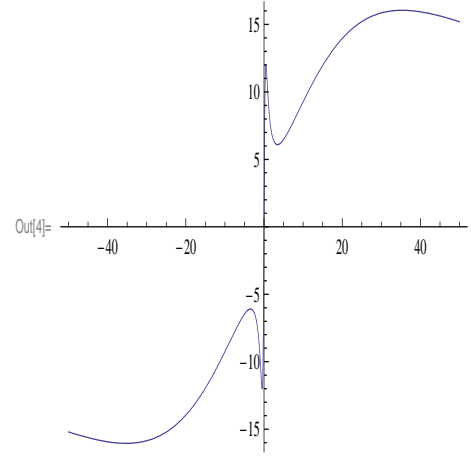
For the set of arbitrary numbers $k_1, ..., k_N$

Fig. 5. Multi q-shock regular for $k_1 = 1$, $k_2 = -1$, $k_3 = 10$, $k_4 = -10$ at $t = 0$

$$\phi(x,t) = \sum_{n=1}^{N} e_q\left(k_n^2 t\right) e_q\left(k_n x\right), \qquad (64)$$

we have multi-shock solution in the form

$$u(x,t) = -2\nu \frac{\sum_{n=1}^{N} k_n e_q\left(k_n^2 t\right) e_q\left(k_n x\right)}{\sum_{n=1}^{N} e_q\left(k_n^2 t\right) e_q\left(k_n x\right)}. \qquad (65)$$

In general this solution admits several singularities. To have regular multi-shock solution we can consider the even number of terms $N = 2k$ with opposite wave numbers. When $N = 4$ and $k_1 = 1$, $k_2 = -1$, $k_3 = 10$, $k_4 = -10$ we have q-multi-shock soliton solution,

$$u(x,t) = -2\nu \frac{e_q(t)\sinh_q(x) + e_q(100t)\sinh_q(10x)}{e_q(t)\cosh_q(x) + e_q(100t)\cosh_q(10x)}. (66)$$

In Fig. 5 we plot $N = 4$ case with values of the wave numbers $k_1 = 1$, $k_2 = -1$, $k_3 = 10$, $k_4 = -10$ at $t = 0$. To have regular solution for any time t and given base q, we should choose proper numbers k_i which are not in the form of power of q. This question is under the study now.

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