Resonant Solitons and Integrable Non-Madelung Fluid Systems

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Abstract:
It is shown that the real version of the second AKNS flow as the reaction-diffusion system, in addition to the Madelung hydrodynamic form written to the current velocity, admits the second hydrodynamic form. This form written for the forward and the backward drift velocity is the Broer-Kaup hydrodynamic system, equivalent to the classical Bousinesque system. By using bilinear form and dissipaton solutions for RD system, we construct solitons for the Broer-Kaup system in non-Burgers reduced form. Resonant dynamics of solitons in BK system is shown and geometrical interpretation in terms of black holes on pseudo-Riemannian surface is given. Then we show that NLS equation also admits new hydrodynamic representation wich can be considered as quantum potential deformation of the BK system. Relations with system of equations including Hamilton-Jacobi-Bellman equation from stochastic control theory and its quantum potential deformation are shown.

Keywords: soliton, resonance, integrable system, hydrodynamics, NLS

1. INTRODUCTION

In 1926, E. Madelung in his first short communication on hydrodynamic interpretation of the Schrodinger equation, introduced complex velocity field

$$\mathbf{V} = \frac{\hbar}{m}\frac{grad\,\psi}{\psi} = (\mathbf{v}_q + i\mathbf{v}_c), \tag{1}$$

with the imaginary part interpreted as velocity of the electron. The last one coincides with the Madelung-Landau-London definition of the local mean velocity

$$\mathbf{v}_c = \frac{i\hbar}{2m\rho}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi) = \frac{\hbar}{m}\Im\frac{\nabla\psi}{\psi} = \frac{\nabla S}{m}. \tag{2}$$

From the Madelung representation, following basic principles of analytical mechanics we can construct most general non-relativistic and nonlinear dispersion. In one space dimension with $\psi = \sqrt{\rho}e^{iS}$ it gives the Cole-Hopf type substitution

$$\frac{\hbar}{m}\frac{\psi_x}{\psi} = \frac{1}{2}(\ln\rho)_x + iS_x \equiv V \tag{3}$$

for complex velocity $V = v_q + iv_c$, where $v_c = \frac{\hbar}{m}S_x$ - is the wave packet velocity or the current velocity of the stochastic mechanics, $v_q = \frac{\hbar}{2m}(\ln\rho)_x$ - internal velocity or the osmotic velocity of the stochastic mechanics (the quantum velocity). Then dispersive part of the energy density is just the quadratic form

$$\epsilon_0 = \frac{\hbar^2}{2m}\bar{\psi}_x\psi_x = \frac{m}{2}\rho(v_c^2 + v_q^2). \tag{4}$$

To construct most general extension of the dispersion, we can follow the postulate of analytical mechanics, where the kinetic term

$$T = \frac{1}{2}\sum_{ij} a_{ij}(q)\dot{q}_i\dot{q}_j \tag{5}$$

is determined by symmetric, positive definite quadratic form a_{ij} for generalized velocities, playing role of the Riemannian metric. Then we have the general dispersion

$$\epsilon_0 = \frac{m}{2}\rho(Ev_c^2 + 2Fv_cv_q + Gv_q^2) \tag{6}$$

in terms of two dimensional metric, or the first fundamental form : (E, F, G). Rewritten in terms of the wave function it gives nonlinear dispersive terms

$$\epsilon_0 = \frac{\hbar^2}{8m}[2E\bar{\psi}_x\psi_x + G|\psi|_x|\psi|_x - \tag{7}$$

$$(E+2iF)\psi_x^2\frac{\bar{\psi}}{\psi} - (E-2iF)\bar{\psi}_x^2\frac{\psi}{\bar{\psi}}]. \tag{8}$$

Here, in addition to standard linear dispersion, the second term represents the quantum potential contribution in form of the Fisher measure. The Nonlinear Schrödinger equations with such term has been considered in Pashaev et al. (2002a), Pashaev et al. (2002b), Lee et al. (2007), and for the last two terms see, Pashaev et al. (2008). For description of envelope soliton resonances, in these papers an extension of metric to the pseudo-Riemannian case and related with it a novel integrable version of NLS equation,

$$i\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + \frac{\Lambda}{4}|\psi|^2\psi = s\frac{1}{|\psi|}\frac{\partial^2|\psi|}{\partial x^2}\psi \tag{9}$$

were introduced, Pashaev et al. (2002a). This has been termed the *Resonant Nonlinear Schrodinger* (RNLS) equation. It can be considered as a third version of the NLS equation, intermediate between the defocusing and focusing cases. Even though the RNLS model is integrable for arbitrary values of the coefficient s, the critical value $s = 1$ separates two distinct regions of behavior. It turns out that for $s < 1$ the model reduces to conventional NLS equation. However, for $s > 1$ it can not be reduced to the usual NLS equation, but rather to a reaction-diffusion (RD) system. In this case the model exhibits the resonance soliton phenomena, Pashaev et al. (2002a).

The Resonant NLS equation (9) admits the Madelung hydrodynamic representation for any s and in this Madelung form with $s > 1$ it appears in plasma physics, Lee et al. (2007). In the present paper we like to show that in the case $s > 1$ the RNLS (9) admits one more hydrodynamic representation in the form of the Broer-Kaup and the classical Boussinesq system, Broer (1975), Kaup (1975). It allows us construct solutions of the last systems in terms of bilinear form for RNLS. We show that soliton solutions for this system have resonance dynamics. Geometric interpretation of hydrodynamic representation and black hole type structure is found. For $s < 1$, when the model is reducible to the standard NLS, we find new hydrodynamic representation in the form of Broer-Kaup system modified by quantum potential term.

2. MADELUNG HYDRODYNAMIC REPRESENTATION AND COLD PLASMA

For $s > 1$, by $\psi = e^{R-iS}$, in terms of $e^+ = e^{R+S}$, $-e^- = e^{R-S}$ after some rescalings we have the system

$$-\frac{\partial e^+}{\partial t} + \frac{\partial^2 e^+}{\partial x^2} + \frac{\Lambda}{4} e^+ e^- e^+ = 0, \tag{10}$$

$$+\frac{\partial e^-}{\partial t} + \frac{\partial^2 e^-}{\partial x^2} + \frac{\Lambda}{4} e^+ e^- e^- = 0. \tag{11}$$

The Madelung form of this system

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{12}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} + \beta^2 \frac{\partial}{\partial x}\left[\frac{1}{\rho}\frac{\partial^2 \rho}{\partial x^2} - \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial x}\right)^2\right] = 0 \tag{13}$$

describes propagation of long magneto-acoustic waves in a cold plasma of density ρ moving with velocity u across the magnetic field, Lee et al. (2007). In this system the dispersion is negative, i.e., the wave velocity decreases with increasing wave vector k.

3. INTERACTING HEAT EQUATION

The system (10),(11) implies the conservation law

$$(e^+ e^-)_t = (e^+_x e^- - e^+ e^-_x)_x. \tag{14}$$

By introducing density $\rho = -e^+ e^-$ it can be rewritten as

$$\rho_t = (\rho(\ln e^+)_x - \rho(\ln e^-)_x)_x. \tag{15}$$

By using identities $\rho_x = \rho(\ln e^+)_x + \rho(\ln e^-)_x$, $\rho_{xx} = (\rho(\ln e^+)_x + \rho(\ln e^-)_x)_x$, and above continuity equation (15) we can get continuity equations for ρ which includes e^+ or e^- field only. Then combining with (10) or (11) we have independent systems for e^+ and ρ

$$-e^+_t + e^+_{xx} - \frac{\Lambda}{4}\rho e^+ = 0 \tag{16}$$

$$\rho_t + \rho_{xx} = (2\rho(\ln e^+)_x)_x \tag{17}$$

and for e^- and ρ

$$+e^-_t + e^-_{xx} - \frac{\Lambda}{4}\rho e^- = 0 \tag{18}$$

$$-\rho_t + \rho_{xx} = (2\rho(\ln e^-)_x)_x \tag{19}$$

The first system describes the heat equation interacting with "potential" ρ and continuity equation, while the second one is the system for the backward heat equation. These two systems are related by time reversing transformation $t \to -t$ and $e^\pm \to e^\mp$.

The system (16) admits reduction $\rho \equiv 0$, so that it reduces to the heat equation. Then simplest solution of the heat equation implies solution of the system (16),(17)

$$e^+ = e^{\frac{1}{4}v^2 t - \frac{1}{2}vx}, \;\; \rho = 0. \tag{20}$$

It should be compared with the next two-parameteric solution, Pashaev et al. (2002a)

$$e^+ = \left(\frac{8}{-\Lambda}\right)^{1/2} \frac{k e^{k^2 t}}{\cosh k(x - vt - x_0)}\, e^{\frac{1}{4}v^2 t - \frac{1}{2}vx}, \tag{21}$$

$$\rho = \frac{8}{-\Lambda}\frac{k^2}{\cosh^2 k(x - vt - x_0)}. \tag{22}$$

Here density ρ is the traveling wave soliton, while for e^+ in moving frame $\xi = x - vt - x_0$ we have the dissipative soliton, *dissipaton*, Pashaev (1997) , with time dependent amplitude

$$e^+ = \left(\frac{8}{-\Lambda}\right)^{1/2} k\frac{e^{[(-\frac{1}{4}v^2 + k^2)t - \frac{1}{2}v(\xi - \xi_0)]}}{\cosh k\xi}. \tag{23}$$

Depending on velocity, it grows with time exponentially when $|v| < 2|k|$, and decays for $|v| > 2|k|$. In the critical case $|v| = 2|k|$ it produces one parametric kink soliton

$$e^+ = \left(\frac{8}{-\Lambda}\right)^{1/2} \frac{k^2}{2}[1 + \tanh\frac{k}{2}(x + kt - x_0)] \tag{24}$$

and

$$\rho = \left(\frac{8}{-\Lambda}\right)\frac{k^2}{4\cosh^2\frac{k}{2}(x + kt - x_0)}. \tag{25}$$

Depending on sign of k it moves to the left if $k > 0$, and to the right if $k < 0$. Similarly, the backward heat equation admits solution

$$e^- = -e^{-(\frac{1}{4}v^2 t - \frac{1}{2}vx)}, \;\; \rho = 0. \tag{26}$$

It should be compared with two parametric solution

$$e^- = -\left(\frac{8}{-\Lambda}\right)^{1/2} \frac{k e^{-k^2 t}}{\cosh k(x - vt - x_0)}\, e^{-(\frac{1}{4}v^2 t - \frac{1}{2}vx)}, \tag{27}$$

$$\rho = \frac{8}{-\Lambda}\frac{k^2}{\cosh^2 k(x - vt - x_0)}. \tag{28}$$

This solution for ρ represents a soliton, while for e^- we have again dissipative soliton with time dependent amplitude. For $|v| = 2|k|$ we have one parametric kink

$$e^- = -\sqrt{\frac{8}{-\Lambda}}\frac{k^2}{2}[1 + \tanh\frac{k}{2}(x - kt - x_0)] \tag{29}$$

and

$$\rho = \left(\frac{8}{-\Lambda}\right) \frac{k^2}{4\cosh^2 \frac{k}{2}(x - kt - x_0)}. \quad (30)$$

Depending on sign of k it moves to the left if $k < 0$, and to the right if $k > 0$.

4. HAMILTON-JACOBI-BELLMAN TYPE REPRESENTATION

Deviding (16) on e^+ and introducing the nonlinear change of variables, similar to the one made first by E. Schrödinger in 1926,

$$A^+(x,t) = 2\ln e^+(x,t) \quad (31)$$

from system (16), (17) we get the equivalent one

$$-A^+_t + \frac{1}{2}(A^+_x)^2 + A^+_{xx} - \frac{\Lambda}{2}\rho = 0 \quad (32)$$

$$\rho_t + \rho_{xx} = (\rho A^+_x)_x \quad (33)$$

Equation (32) is well known in the theory of optimal stochastic control for continuous Markov processes, and in this context it is called the "Hamilton-Jacobi-Bellman equation"(HJB). The meaning of the variable change (31) is that equation (32) is a dynamic programming equation whose solution is the minimum value of some action functional. In our case we have the HJB equation (32) coupled with the continuity equation (33). The system (32), (33) can be written as the Euler-Lagrange equation with variational functional $S = \int\int L dx dt$ and the Lagrangian density

$$L = \frac{1}{2}\rho(A^+_t - A^+_{xx}) - \frac{1}{4}\rho(A^+_x)^2 + \frac{\Lambda}{8}\rho^2. \quad (34)$$

For one dissipaton solution (21) we have solution of the system (32),(33) with

$$A^+ = \left(k^2 + \frac{1}{2}v^2\right)t - vx - \quad (35)$$

$$2\ln[\cosh k(x - vt - x_0)] + \ln(\frac{8\sqrt{k}}{-\Lambda}) \quad (36)$$

and (22) for ρ.

Similar procedure for the backward system (18), (19) in terms of function

$$A^-(x,t) = 2\ln(-e^-(x,t)) \quad (37)$$

gives the system

$$A^-_t + \frac{1}{2}(A^-_x)^2 + A^-_{xx} - \frac{\Lambda}{2}\rho = 0 \quad (38)$$

$$-\rho_t + \rho_{xx} = (\rho A^-_x)_x \quad (39)$$

5. BROER-KAUP HYDRODYNAMIC REPRESENTATION

The heat equation is naturaly connected with the Burgers equation by the Cole-Hopf transformation. This implies introduction of velocity field $v^+ = (\ln e^+)_x$ so that the system (16),(17) becomes

$$v^+_t = (v^+_x + {v^+}^2 - \frac{\Lambda}{4}\rho)_x, \quad (40)$$

$$\rho_t + \rho_{xx} = (2\rho v^+)_x. \quad (41)$$

This system is known as the Broer-Kaup system, Broer (1975), Kaup (1975) . For $v^- = (\ln e^-)_x$ the system (18),(19) implies

$$-v^-_t = (v^-_x + {v^-}^2 - \frac{\Lambda}{4}\rho)_x \quad (42)$$

$$-\rho_t + \rho_{xx} = (2\rho v^-)_x \quad (43)$$

- the anti Kaup-Broer system. For $\rho = 0$ the systems can be reduced to the Burgers equation and to the anti-Burgers equations correspondingly.

For $\rho \neq 0$ from dissipaton solution (21) we get the kink soliton

$$v^+ = -\frac{v}{2} - k\tanh k(x - vt - x_0) \quad (44)$$

for (40) and from dissipaton solution (27) we get the anti-kink soliton

$$v^- = \frac{v}{2} - k\tanh k(x - vt - x_0) \quad (45)$$

for (42).

6. CLASSICAL BOUSSINESQ REPRESENTATION

Equation (40) can be represented in the Euler form. If we introduce the "pressure" function

$$p^+ = -\frac{\Lambda}{4}\rho + v^+_x \quad (46)$$

then from (40), (41) we have the classical Boussinesq system

$$-v^+_t + 2v^+v^+_x = -p^+_x, \quad (47)$$

$$p^+_t = v^+_{xxx} + (2p^+v^+)_x. \quad (48)$$

Similar way by introduction

$$p^- = -\frac{\Lambda}{4}\rho + v^-_x \quad (49)$$

from (42), (43) we have the classical anti-Boussinesq system

$$v^-_t + 2v^-v^-_x = -p^-_x, \quad (50)$$

$$-p^-_t = v^-_{xxx} + (2p^-v^-)_x. \quad (51)$$

7. BILINEAR FORM AND SOLITONS

Representing two real functions e^+, e^- in terms of three real functions g^+, g^-, and f,

$$e^\pm = \sqrt{\frac{8}{-\Lambda}}\frac{g^\pm}{f} \quad (52)$$

we have the next bilinear system of equations

$$(\pm D_t - D^2_x)(g^\pm \cdot f) = 0, \quad (53)$$

$$D^2_x(f \cdot f) = -2g^+g^-. \quad (54)$$

Any solution of (53),(54) with

$$\rho = \left(\frac{8}{\Lambda}\right)\frac{g^+ g^-}{f^2} = \left(\frac{8}{\Lambda}\right)(\ln f)_{xx} \tag{55}$$

gives solution of the system (16),(17) and (18),(19).

Solution of the Broer-Kaup system (40),(41) is

$$v^+ = \frac{D_x(g^+\cdot f)}{g^+ f} = \left(\ln\frac{g^+}{f}\right)_x, \tag{56}$$

while for system (42),(43)

$$v^- = \frac{D_x(g^-\cdot f)}{g^- f} = \left(\ln\frac{g^-}{f}\right)_x. \tag{57}$$

Solution of the classical Boussinesq system (47), (48) is

$$p^+ = (\ln(g^+ f))_x,\ \ v^+ = \left(\ln\frac{g^+}{f}\right)_x, \tag{58}$$

while for (50), (51) it is

$$p^- = (\ln(g^- f))_x,\ \ v^- = \left(\ln\frac{g^-}{f}\right)_x. \tag{59}$$

7.1 One Soliton Solution

The one-dissipaton solution of (53),(54) is given by : $g^\pm = \pm e^{\eta_1^\pm}$, $f = 1 + e^{\eta_1^+ + \eta_1^- + \phi_{1,1}}$, $e^{\phi_{1,1}} = (k_1^+ + k_1^-)^{-2}$, where $\eta_1^\pm \equiv k_1^\pm x \pm (k_1^\pm)^2 t + \eta_1^{\pm(0)}$ and $k_1^\pm$, $\eta_1^{\pm(0)}$ are constants.

It gives kink-soliton solution for v^+ (40),(41)

$$v^+ = \frac{k_1^+ - k_1^-}{2} - \frac{k_1^+ + k_1^-}{2}\tanh\frac{\eta_1^+ + \eta_1^- + \phi_{11}}{2} \tag{60}$$

and soliton shape for density ρ (Fig1., Fig 2.)

$$\rho = \frac{2}{-\Lambda}\frac{(k_1^+ + k_1^-)^2}{\cosh^2\frac{\eta_1^+ + \eta_1^- + \phi_{11}}{2}} \tag{61}$$

In terms of parameters, $k \equiv (k_1^+ + k_1^-)/2$, $v \equiv -(k_1^+ - k_1^-)$ it gives solution (44). For the system (42),(43) it gives the anti-kink solution (45).

For the classical Boussinesq system (47),(48) we have kink-soliton

$$p^+ = (2k - \frac{v}{2}) + k\tanh k(x - vt - x_0) \tag{62}$$

while for anti-Boussinesq (50),(51), we get anti-kink

$$p^- = (2k + \frac{v}{2}) + k\tanh k(x - vt - x_0). \tag{63}$$

For two-dissipaton solution we have

$$g^\pm = \pm[e^{\eta_1^\pm} + e^{\eta_2^\pm} + (\frac{\breve{k}_{12}^{\pm\pm}}{k_{21}^{\pm\mp}k_{11}^{+-}})^2 e^{\eta_1^+ + \eta_1^- + \eta_2^\pm} + (\frac{\breve{k}_{12}^{\pm\pm}}{k_{12}^{\pm\mp}k_{22}^{+-}})^2 e^{\eta_2^+ + \eta_2^- + \eta_1^\pm}]$$

$$f = 1 + \frac{e^{\eta_1^+ + \eta_1^-}}{(k_{11}^{+-})^2} + \frac{e^{\eta_1^+ + \eta_2^-}}{(k_{12}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_1^-}}{(k_{21}^{+-})^2} + \frac{e^{\eta_2^+ + \eta_2^-}}{(k_{22}^{+-})^2} + (\frac{\breve{k}_{12}^{++}\breve{k}_{12}^{--}}{k_{12}^{+-}k_{21}^{+-}k_{11}^{+-}k_{22}^{+-}})^2 e^{\eta_1^+ + \eta_1^- + \eta_2^+ + \eta_2^-},$$

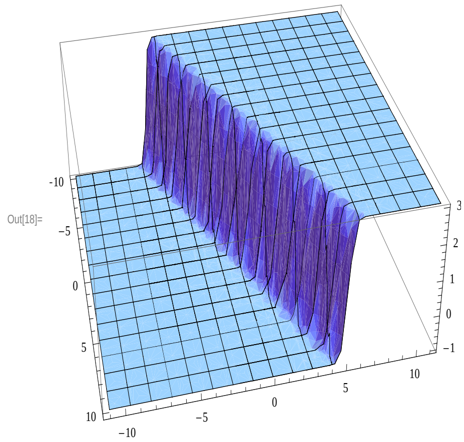

Fig. 1. Domain wall shape for velocity field.

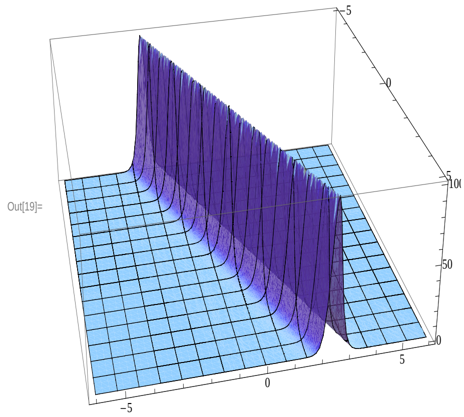

Fig. 2. Soliton shape for density field.

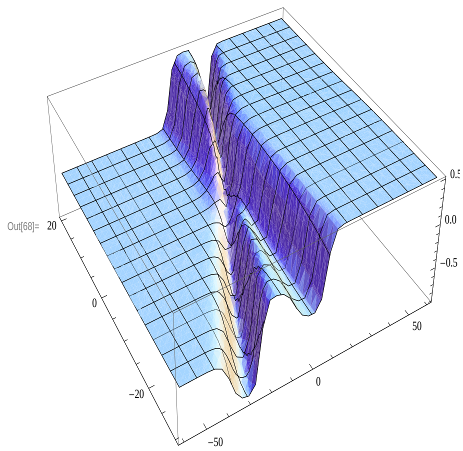

Fig. 3. Two soliton velocity field, $k_1^+ = 0.5$, $k_1^- = 0.9$, $k_2^+ = 0.1$, $k_2^- = 0.35$

where $k_{ij}^{ab} \equiv k_i^a + k_j^b$, $\breve{k}_{ij}^{ab} \equiv k_i^a - k_j^b$, $\eta_i^\pm \equiv k_i^\pm x \pm (k_i^\pm)^2 t + \eta^{\pm(0)}$. This solution shows the resonance character of dissipaton interaction, Pashaev et al. (2002a)(Fig3., Fig4.)

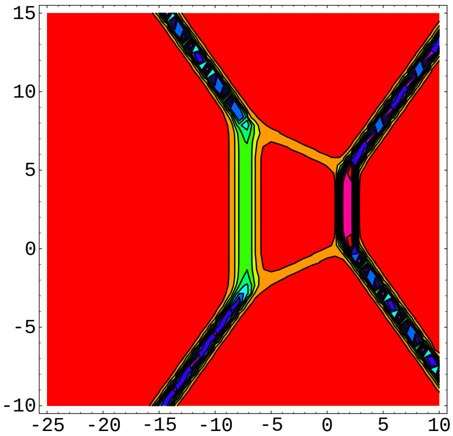

Fig. 4. Four soliton resonance for the density field, $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 1$, $k_2^- = 2$, d=16

7.2 Burgers reduction

Velocity v^+ for BK system in Hirota form is (57)

$$v^+ = \frac{g_x^+}{g^+} - \frac{f_x}{f} \tag{64}$$

If $g^+ = const$, then $v^+ = -(\ln f)_x$ which implies constraint

$$v_x^+ = \frac{\Lambda}{8}\rho. \tag{65}$$

Under this reduction the BK system reduces to the Burgers equation

$$v_t^+ - 2v^+ v_x^+ + v_{xx}^+ = 0. \tag{66}$$

Then the first bilinear equation (54) reduces to the heat equation

$$f_t - f_{xx} = 0 \tag{67}$$

and the above form $v^+ = -(\ln f)_x$ is just the Cole-Hopf transformation. On the level of one soliton solution with $k_1^+ = 0$ it gives just Burgers shock soliton solution. Then, for two dissipaton case, puting $k_1^+ = 0$, $k_2^+ = 0$ we get two shock solitons

$$v^+ = -\frac{\frac{2}{k_1^-}e^{\eta_1^-} + \frac{2}{k_2^-}e^{\eta_2^-}}{1 + \frac{2}{(k_1^-)^2}e^{\eta_1^-} + \frac{2}{(k_2^-)^2}e^{\eta_2^-}}. \tag{68}$$

This solution describes fusion of two shock solitons. Another reduction $g^- = const$, when $v^- = -(\ln f)_x$, implies

$$v_x^- = \frac{\Lambda}{8}\rho. \tag{69}$$

Under this reduction the BK system reduces to the backward Burgers equation

$$-v_t^- - 2v^- v_x^- + v_{xx}^- = 0. \tag{70}$$

Two shock soliton solutions in this case describe decay of one soliton on two.

8. INTEGRALS OF MOTION

Here we find integrals of motion for the Broer-Kaup system (40),(41) under the vanishing boundary conditions $\rho \to 0$ when $|x| \to \infty$. The first integral of motion is the mass and it follows just from the second equation of the system (41)

$$M = \int_{-\infty}^{\infty} \rho\, dx. \tag{71}$$

The momentum conservation law

$$(\rho v^+)_t = (\rho v_x^+ - \rho_x v^+ + 2\rho {v^+}^2 - \frac{\Lambda}{8}\rho^2)_x \tag{72}$$

implies the momentum integral

$$P^+ = -2\int_{-\infty}^{\infty} \rho v^+\, dx. \tag{73}$$

The energy conservation law

$$(\rho {v^+}^2 - \rho_x v^+ - \frac{\Lambda}{8}\rho^2)_t =$$

$$(-3\rho_x {v^+}^2 + 2\rho {v^+}^3 + \rho_{xx} v^+ - \rho_x v_x^+ - \frac{\Lambda}{2}\rho^2 v^+ + \frac{\Lambda}{4}\rho\rho_x)_x$$

gives the energy integral

$$E^+ = 2\int_{-\infty}^{\infty} (\rho {v^+}^2 + \rho v_x^+ - \frac{\Lambda}{8}\rho^2)\, dx. \tag{74}$$

In a similar way possible to treat the anti-Broer-Kaup system (42),(43), so we have integrals

$$P^- = 2\int_{-\infty}^{\infty} \rho v^-\, dx, \tag{75}$$

$$E^- = 2\int_{-\infty}^{\infty} (\rho {v^-}^2 + \rho v_x^- - \frac{\Lambda}{8}\rho^2)\, dx. \tag{76}$$

Under Hirota's substitution we have

$$M = \int_{-\infty}^{\infty} \rho\, dx = \frac{8}{-\Lambda}\int_{-\infty}^{\infty} (\ln f)_{xx} dx = \frac{8}{-\Lambda}(\ln f)_x|_{-\infty}^{\infty} \tag{77}$$

For one-soliton solution we have the mass

$$M = \frac{8}{-\Lambda}|k_1^+ + k_1^-|. \tag{78}$$

For momentum we have

$$P^+ = \frac{8}{-\Lambda}|k_1^+ + k_1^-|(k_1^- - k_1^+) = Mv, \tag{79}$$

where $v = k_1^- - k_1^+$.

9. GEOMETRICAL REPRESENTATION

We introduce pseudo-Riemannian metric in terms of the drift velocities, Pashaev et al. (2002a)

$$g_{00} = \rho v^+ v^-, \quad g_{11} = -\rho, \quad g_{01} = \frac{1}{2}\rho(v^- - v^+). \tag{80}$$

For RD system this metric describes pseudo-Riemannian surface of constant scalar curvature $R = \Lambda$. At a zero of drift velocity, as a solution of equation $v^+(x,t) = 0$ or $v^-(x,t) = 0$, this metric develops horizon singularity and the black hole type picture.

This metric can be rewritten in terms of the Broer-Kaup hydrodynamics. For (40), (41) we have

$$g_{00} = v^+(\rho_x - \rho v^+), \;\; g_{11} = -\rho, \;\; g_{01} = \frac{1}{2}\rho_x - \rho v^+. \quad (81)$$

At horizon $g_{00} = 0 \to v^+ = 0$ or $\rho_x = \rho v^+$ and

$$\tanh k(x - vt - x_0) = \mp \frac{v}{2k} \quad (82)$$

and for $|v| < 2|k|$ it admits two horizons.

For anti-Broer-Kaup system we have

$$g_{00} = v^-(\rho_x - \rho v^-), \;\; g_{11} = -\rho, \;\; g_{01} = -\frac{1}{2}\rho_x + \rho v^-. \quad (83)$$

10. THIRD RD HIERARCHY FLOW AND HIGHER HYDRODYNAMIC SYSTEM

The RD system is the second flow of $SL(2,R)$ AKNS hierarchy. For the third flow of the hierarchy we have

$$e_t^+ = e_{xxx}^+ + \frac{3\Lambda}{4} e^+ e^- e_x^+, \quad (84)$$

$$e_t^- = e_{xxx}^- + \frac{3\Lambda}{4} e^+ e^- e_x^-. \quad (85)$$

Following similar procedure as for the RD system in terms of hydrodynamic variables $v^+ = (\ln e^+)_x$ and $\rho = e^+e^-$ we have new system

$$v_t^+ = (v_{xx}^+ + 3v^+v_x^+ + (v^+)^3 + \frac{3\Lambda}{4}\rho v^+)_x, \quad (86)$$

$$\rho_t = (\rho_{xx} - 3\rho_x v^+ + 3\rho(v^+)^2 + \frac{3\Lambda}{8}\rho^2)_x. \quad (87)$$

10.1 KdV Reduction

In particular case $v^+ = 0$ it reduces to KdV equation

$$\rho_t = \rho_{xxx} + \frac{3\Lambda}{4}\rho\rho_x. \quad (88)$$

10.2 Modified MKdV Reduction

For particular case $\rho = 0$ this system reduces to the following modification of the MKdV equation

$$v_t^+ = (v_{xx}^+ + 3v^+v_x^+ + (v^+)^3)_x \quad (89)$$

$$= v_{xxx}^+ + 3(v^+)^2 v_x^+ + (3v^+v_x^+)_x \quad (90)$$

10.3 Bilinear Form

By substitution $e^{\pm} = \sqrt{\frac{8}{-\Lambda}} \frac{g^{\pm}}{f}$ we get

$$(D_t + D_x^3)(g^{\pm} \cdot f) = 0, \quad (91)$$

$$D_x^2(F \cdot F) = -2g^+g^-. \quad (92)$$

Then solution of the system is given by

$$v^+ = \frac{g_x^+}{g^+} - \frac{f_x}{f}, \quad \rho = \frac{8}{\Lambda}\frac{\partial^2}{\partial x^2}\ln F. \quad (93)$$

For one soliton solution we have kink

$$v^+ = \sqrt{\frac{4v^2 - k^2}{12}} - \frac{k}{2}\tanh\frac{k}{2}(x - vt - x_0) \quad (94)$$

and soliton

$$\rho = \frac{8}{4\Lambda}\frac{k^2}{\cosh^2\frac{k}{2}(x - vt - x_0)}, \quad (95)$$

where $k = k_1^+ + k_1^-$, $v = (k_1^+)^2 - k_1^+k_1^- + (k_1^-)^2$. The last relation implies restriction on speed of soliton $|v| > |k|/4$.

11. BROER-KAUP SYSTEM WITH QUANTUM POTENTIAL

In this section we show that NLS equation can be represented as new Broer-Kaup hydrodynamic system with quantum potential. This representation is alternative to the known Madelung representation of NLS. If in NLS equation

$$i\psi_t + \psi_{xx} + \frac{\Lambda}{4}|\psi|^2\psi = 0 \quad (96)$$

by Madelung substitution $\psi = e^{R-iS}$ we introduce density $\rho = |\psi|^2$, center of mass or current velocity $V = -2S_x$ and quantum or osmotic velocity $V_Q = 2R_x$, then in terms of ρ and V one can get the Madelung fluid form of NLS. In contrast here we introduce the drift velocities $v^+ = (V_Q - V)/2$ and $v^- = (V_Q + V)/2$. Then we can rewrite NLS equation in terms of v^+ and ρ as

$$v_t^+ = (v_x^+ + {v^+}^2 - \frac{\Lambda}{4}\rho - 2\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}})_x, \quad (97)$$

$$\rho_t + \rho_{xx} = (2\rho v^+)_x, \quad (98)$$

and in terms of v^- and ρ as

$$-v_t^- = (v_x^- + {v^-}^2 - \frac{\Lambda}{4}\rho - 2\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}})_x, \quad (99)$$

$$-\rho_t + \rho_{xx} = (2\rho v^-)_x. \quad (100)$$

These new equations can be considered as a quantized Broer-Kaup systems.

REFERENCES

Pashaev O.K. and Lee J.-H., *Mod. Phys. Lett. A*, Vol. 17, No. 24 ,(2002), pp. 1601-1619.

Pashaev O K and Lee J H, *ANZIAM J.*(2002) 44 7381

Lee J.-H, Pashaev O.K., Rogers C. and Schieff W., *Journal of Plasma Physics*, V. 73, (2007), 257-272.

Pashaev O.K., Lee J.-H. and Rogers C., *Journal of Physics. A: Mathematical and Theoretical*, Vol. 41, No. 24 ,(2008), 452001.

Pashaev O.K., *Nuclear Physics B (Proc. Suppl.)*, Vol. 57,(1997), pp. 338-341.

Broer L. J. F., Approximate equations for long water waves, *Appl. Sci. Res.* 31 (1975) 377

Kaup D. J., A higher-order water- wave equation and the method for solving it, *Progr Theo Phys* 54 (1975) 396