Evidence of the Wave Phase Coherence for Freak Wave Events ⋆

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Abstract: In this paper we demonstrate the strong correlation in the spectrum area close to the spectral peak in cases when the Benjamin - Feir instability causes intense wave groups of unidirectional deep-water surface water waves referred to freak events. A simple phase coherence estimator in the form of an autobicorrelation function is suggested and tested on the basis of the results of numerical simulations within different frameworks. The correlation reaches the value of a unity, and, thus, the random phase approximation is definitely violated for these waves.

Keywords: Rogue waves, nolinear waves, phase coherence.

1. INTRODUCTION

Irregular waves arise in many important physical problems and are subject of investigation. The study of irregular wave dynamics and statistics is relevant for correct physical understanding and for practical applications as well. Sea waves are an example of inherently stochastic waves. They are often understood as a combination of quasi-sinusoidal waves with independent random uniformly distributed phases (the Gaussian sea). If waves were linear and random, they would possess the Gaussian probability distribution function due to the central limit theorem.

The difference between the Gaussian sea approximation and the real sea results in recognizing the problem of *freak wave* or *rogue wave* phenomenon, see reviews (18; 11; 19). The attempts to integrate the wave nonlinearity effect into the statistical models have been undertaken many times with a certain success. However, each time these endevours employ the condition of weak nonlinearity, which is questionable when applied to freak wave events. Meanwhile, obtaining the statistical description of freak waves for the case of a given wave energy spectrum is the cherished aim of the researchers.

Kinetic approach is conventional and well-established for the study of random wave spectrum evolution. The kinetic theory is weakly nonlinear, uses closure assumptions, thus, is eventually capable of describing only near-Gaussian processes, and it disregards wave phases. The stochastic approach employs dynamical models which resolve the wave phases. Then the relations between the spectral and statistical wave characteristics may be established. Instead of computing the kinetic equations, the stochastic approach has become very popular due to computer power progress and building large and well-equipped experimental facilities.

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The stochastic modelling requires definition of the initial wave fields. Typically the initial condition is defined in the form of wave fields obeying some spectrum with random uniformly distributed phases. A great number of numerical and laboratory experiments prove that this initial condition undergoes strong evolution at the initial stage, what changes the average spectrum of the waves, see (21; 22; 13; 15; 23; 31; 20; 4; 5; 6; 24; 26; 27; 28) among others. Thus, the wave fields at this stage cannot be considered statistically stationary. Limited fetches (due to the limitation of the laboratory facility or short numerical simulations) may prevent the achievement of the stationary state at all.

The most striking nonlinear effect, which is now believed able to cause rogue waves is the Benjamin - Feir instability (otherwise, the side-band or the modulational instability, see for instance (16; 34)), which leads to the generation of intense wave groups from uniform wave trains in deep-water conditions. This effect for unidirectional waves has been confirmed many times by means of numerical simulations, and also by laboratory measurements. Meanwhile, it is known to be weakened and even cancelled, when broadband waves or random waves are concerned.

The Benjamin - Feir index (BFI) was introduced by Onorato et al. (21) and Janssen (15) to measure the strength of the nonlinear self-modulation effects for a given wave energy spectrum. This compound spectral parameter is in agreement with its dynamic counterpart, which follows from the weakly nonlinear weakly modulated theory for surface waves (the framework of the nonlinear Schrödinger equation, NLS). Then the BFI corresponds to the similarity parameter of the NLS equation, otherwise, the soliton number.

The NLS equation is a unique mathematical model due to the property of integrability. Its solutions have been suggested to describe real freak waves in the ocean (12; 14; 25; 1). The so-called *breather* solutions of the NLS equation are actually solitary waves interacting with other background waves. The similarity of the breather solutions and the large-wave events observed in numerical simulations has been pointed out many times (14; 8; 7). The existence of long-living strongly nonlinear wave groups similar to the envelope solitons has been reported recently for numerical simulations of fully nonlinear equations for hydrodynamics (9; 29).

The framework of the NLS equation for unidirectional waves requires the conditions of weak nonlinearity and narrow spectrum. In the general case the solitary-like patterns are supposed to show themselves as coherent wave structures, what implies non-zero correlation between the Fourier modes. The dynamics of a single four-wave resonance quartet was studied within the framework of the Zakharov equation by means of the analytic solution in (32; 17); in (32) ensembles of the wave quartets were considered as well and compared versus the results of the kinetic approach (15). It was shown, in particular, that initially random phases develop a significant coherence in the course of evolution.

In this paper we deal with the deep-water limit, and the unidirectional case of surface sea waves, which is the most investigated. The wave trains are generally supposed narrow-band; the physics is governed by the free wave components, while the bound waves (Stokes corrections) under the narrow-band assumption may be trivially obtained on the basis of the free waves.

The coherence might be revealed in the dynamics of the strongly interacting components (when they may be singled out) such as resonance quartets or soliton-like wave groups. We show that the wave coherence can be revealed globally in a stochastic wave field. A more detailed description of this study is given in (30).

2. STOCHASTIC NUMERICAL SIMULATIONS OF MODULATIONALLY UNSTABLE WAVE FIELDS

In this section we summarize the results of numerical simulations, performed within the frameworks of the nonlinear Schrödinger equation, its high-order generalization, the Dysthe equation with the exact deep-water linear dispersion law taken into account (10; 33), and the fully nonlinear simulations of the Euler equations in the conformal variables. The algorithms are briefly described in (29). Non-breaking waves were considered; 100 wave ensembles were used for the averaging.

In all cases the initial wave realizations were defined in the form of a linear superposition of Stokes waves with random phases, similar to those described in (27), which obey the Gaussian spectrum. The carrier wavenumber was chosen $k_0 \approx 1.7802$ m^{-1}. Due to the deep-water conditions, the mean wave period is defined through the linear dispersion law as $T_0 = 1.5$ s. The NLS model was used to solve waves with a moderate initial steepness, $k_0\eta_{rms} \approx 0.042$ (where η_{rms} is the root-mean-square surface wave displacement defined on the basis of the free wave component), for the different initial spectrum widths. The fully nonlinear simulations were performed for one initial spectrum width, $\nu/k_0 \approx 0.076$, where ν is defined as the second moment of the average spectrum. In terms of the introduced parameters, the BFI may be defined as (15)

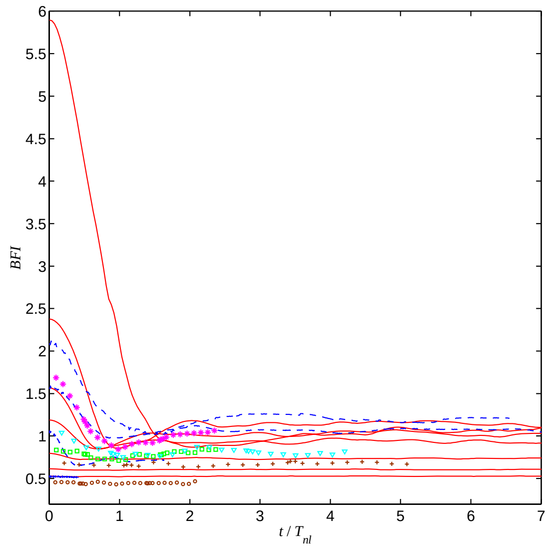


Fig. 1. The temporal dependences of the BFI versus the scaled time for numerical simulations (lines) with different initial spectrum widths and wave intensities, and for the laboratory measurements (28) (symbols) with different spectral widths and shapes

$$BFI = 2\sqrt{2}k_0^2\frac{\eta_{rms}}{\nu}, \tag{1}$$

and the characteristic nonlinear time we define as

$$T_{nl} = \frac{1}{\omega_0 k_0^2 \eta_{rms}^2}. \tag{2}$$

In the course of evolution the spectrum width, ν and the BFI evolve with time. The variable BFI for the simulations is shown in Fig. 1 as the function of the scaled time. The results of the NLS simulations are represented by red solid lines, and the results of the fully nonlinear simulations are represented by blue dashed lines.

The results of laboratory measurements (26; 28) are given in Fig. 1 with symbols. In the laboratory experiments the spatial wave evolution was considered, so that the distances are recomputed to the corresponding times supposing that the waves propagate with the group velocity. It is evident that the curves corresponding to the two sets of numerical simulations and the results of laboratory measurements well agree when represented in the scaled variables. For sufficiently long times kinds of steady states are achieved in all the cases.

The Alber theory for narrow-band weakly nonlinear random waves (2; 15) predicts the cancellation of the Benjamin - Feir instability effect for $BFI < 1$. It may be seen in Fig. 1 that this threshold describes well the qualitative difference in the evolution of the BFI. The wave fields with initially large values of the BFI tend to the state which seems to be marginally stable. The wave fields with small values of the BFI remain practically unchanged during the evolution.

3. THE EVIDENCE OF PHASE COHERENCE

The spectral phases are implied by the concept of the Gaussian sea to be uniformly distributed. Indeed, the phase distribution observed in our simulations may be considered uniform (see (30)). However, the nonlinear wave phases are obviously not fully independent as it is in the linear approximation.

The phase coherence is supported by the presence of coherent wave structures. Some exact solutions of the NLS equations are discussed in this context in (30). When many nonlinear coherent structures are present in the wave field, the dynamics is supposed to be quite complicated.

We apply the following correlation function to make the coherence between the Fourier phases evident:

$$\begin{aligned} R(\delta,t) &= \frac{R_1}{R_2}, \\ R_1 &= \left|\sum_{n=1}^{N} S_n(k_0+\delta)S_n(k_0-\delta)S_n^*(k_0+\delta')S_n^*(k_0-\delta')\right|, \\ R_2 &= \sum_{n=1}^{N} \left|S_n(k_0+\delta)S_n(k_0-\delta)S_n^*(k_0+\delta')S_n^*(k_0-\delta')\right|, \\ \delta &= \frac{2\pi}{L}m, \quad \delta' = \frac{2\pi}{L}(m+1), \quad m \geq 0. \end{aligned} \tag{3}$$

In (3) k_0 is the dominant wavenumber, δ and δ' specify the wavenumber offsets according to the wavenumber discretization (m counts the spectral nodes), L is the computational domain length. The summation is performed over all N realizations.

The results of the computation of the autobicoherence function (3) for the conditions $k_0\eta_{rms} \approx 0.042$, $\nu/k_0 \approx 0.076$ (and $BFI \approx 1.56$) are presented in Fig. 2 by the colour intensity. Different models are simulated: the NLS equation (Fig. 2a), the Dysthe equation (Fig. 2b), and the Euler equations in conformal variables (Fig. 2c). The horizontal axis shows the normalized time, the vertical axis represents the wavenumber offset. The temporal dependence of the average spectrum width ν is given over the diagram (the solid line), and the spectrum shape at $t = 0$ for $k \geq k_0$ is shown to the right of the diagram (the line with circles); both are given for the reference.

The case of the NLS equation simulation is shown in Fig. 2a. It is obvious that the initial condition corresponds to an insignificant correlation (dark area near $t = 0$). But with time the correlation for a sufficiently large offset quickly grows up to unity. While the Fourier modes which are sufficiently far from the spectral peak are thus shown to be absolutely correlated, the most energetic area of the spectrum close to the spectral peak turns out to be uncorrelated (at least with respect to the autobicorrelation function (3)).

The obtained result is not an artifact of the NLS approximation, but is confirmed in the simulations of the Dysthe model (Fig. 2b) as well as the fully nonlinear equations (Fig. 2c). Some difference between the diagrams Fig. 2a-c may be noticed only far from the dominant wavenumber.

Steeper wave conditions ($k_0\eta_{rms} = 0.056$) are concerned in Fig. 3 for the same initial spectrum width $\nu/k_0 \simeq 0.076$ ($BFI = 2.08$). The waves are simulated by means of the

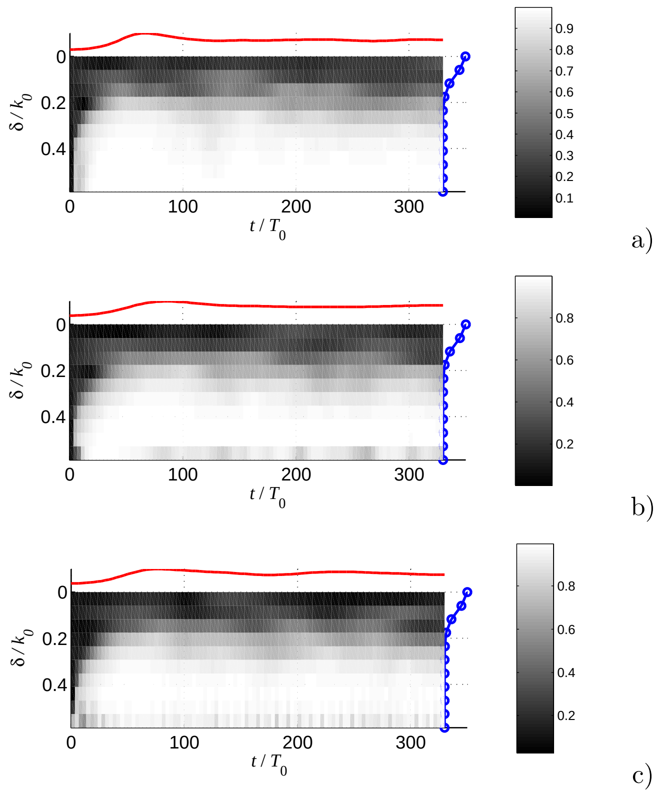


Fig. 2. Diagrams of the correlation estimator $R(\delta,t)$ shown by the colour intensity. The temporal dependence of the average spectrum width is given over the diagram (the solid line) for the reference, and the spectrum shape at $t = 0$ is shown to the right of the diagram (the line with circles) for the reference. The initial condition for simulation $BFI(t = 0) = 1.56$ is computed in different frameworks: the NLS equation (a), the Dysthe model (b), the fully nonlinear simulations (c)

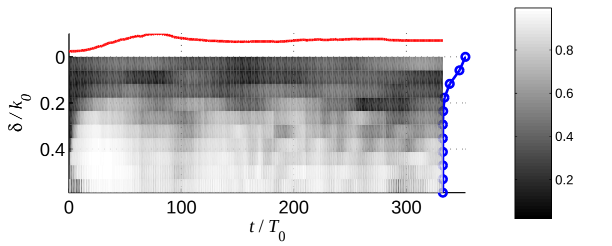
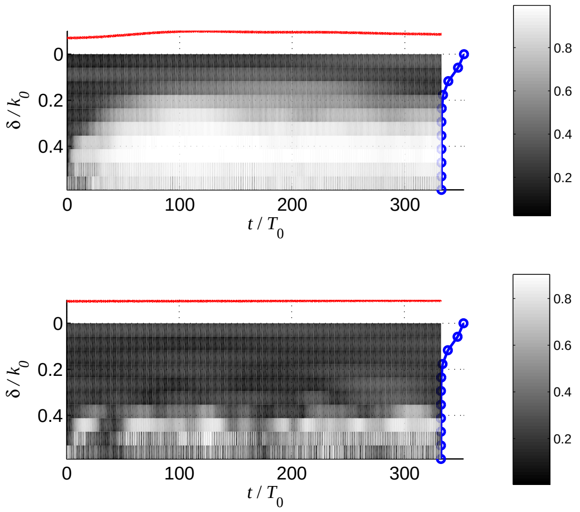


Fig. 3. The diagram of the correlation estimator $R(\delta,t)$ shown by the colour intensity. The temporal dependence of the average spectrum width is given over the diagram (the solid line) for the reference, and the spectrum shape at $t = 0$ is shown to the right of the diagram (the line with circles) for the reference. The fully nonlinear simulation with $BFI(t = 0) = 2.08$ is reported

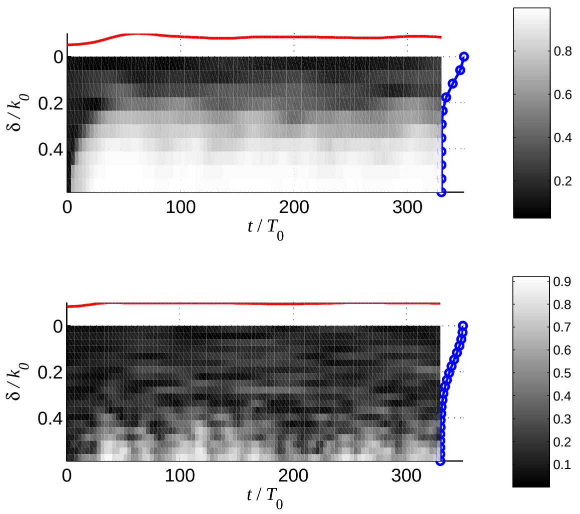
fully nonlinear model. Although the correlation picture in Fig. 3 is less sharp than in Fig. 2, the level of the phase coherence is again very high. It is also significant that in the course of evolution about one half of the realizations resulted in very steep waves; these simulations were not taken into account after the steep event occurrence. Thus,



a)

b)

Fig. 4. Diagrams of the correlation estimator $R(\delta, t)$ shown by the colour intensity. The temporal dependence of the average spectrum width is given over the diagram (the solid line) for the reference, and the spectrum shape at $t = 0$ is shown to the right of the diagram (the line with circles) for the reference. The fully nonlinear simulations with $BFI(t = 0) = 1.04$, (a) and $BFI(t = 0) = 0.52$, (b) are reported (different wave intensities)



a)

b)

Fig. 5. Diagrams of the correlation estimator $R(\delta, t)$ shown by the colour intensity. The temporal dependence of the average spectrum width is given over the diagram (the solid line) for the reference, and the spectrum shape at $t = 0$ is shown to the right of the diagram (the line with circles) for the reference. The NLS equation simulations with $BFI(t = 0) = 1.19$, (a) and $BFI(t) = 0) = 0.79$, (b) are reported (different spectrum widths)

in a certain sense, less coherent wave fields compose the statistical data at longer times.

Different wave amplitudes with the same initial spectrum width are considered in Fig. 4 within the fully nonlinear framework with the initial values of the BFI equal to 1.04 and 0.52 (Fig. 4a and Fig. 4b respectively). The level of the correlation and the interval of wave numbers, where the coherence is revealed are noticeably smaller for the case displayed in Fig. 4b in comparison with Fig. 4a.

In Fig. 5 wave fields with the same root-mean-square surface displacement but different initial spectrum widths are considered. The typical wave amplitude is modest, and the NLS equation is solved to obtain the results. Similar to Fig. 4, the initial BFIs were chosen equal to 1.19 and 0.79 in Fig. 5a and Fig. 5b correspondingly. Again, the coherence is much less evident, if $BFI < 1$ (Fig. 5b), than if $BFI > 1$ (Fig. 5a).

4. CONCLUSION

The stochastic approach to the study of intense sea waves, and in particular the freak or rogue waves, has become quite popular during the recent years. The irregular waves are usually defined as a linear superposition of Stokes waves with random phases following the conventional concept of a Gaussian or near-Gaussian sea. Although the random phase assumption is acknowledged to be violated due to nonlinearity, the obtaining of the phase correlation functions for realistic waves from general principles is a hard task. The four-wave interactions are the most efficient for the deep-water case, but the resonance wave quartets interact among each other (32; 17); moreover, quasi-resonance wave interactions obviously have to be taken into account (32; 3) to obtain a reliable result.

Non-resonant interactions lead to the appearance of phase-locked modes in the Fourier spectrum. The bound wave components (otherwise the nonlinear Stokes corrections) at multiple frequencies/wavenumbers are naturally observed in experiments and may be accounted by the accepted models. The free wave components (exact resonant or near resonant wave modes) are considered governing the physics of nonlinear waves. The bound waves at multiple frequencies/wavenumbers are an indicator of occurrence of steep waves. The bound waves are coherent waves, but in the spectrum they are far from the spectral peak.

The Benjamin - Feir (modulational) instability has been shown to be a nonlinear effect which increases the probability of freak waves in the deep-water narrow-band case. The onset of the Benjamin - Feir nonlinear instability is well controlled by the Benjamin - Feir index, BFI. Initially unstable wave fields tend to a marginally stable state. The self-modulation effect becomes apparent through the occurrence of large-amplitude coherent wave groups.

In this short note we employ the empirically written estimator for the wave coherence, complied with exact model solutions of the nonlinear Schrödinger equation as well as with the general comprehension of the resonance wave quartet nature, see details in (30). We focus the attention on the coherence in the Fourier space, which corresponds to harmonics, phase-locked due to the nonlinearity. These modes indicate the presence of the mentioned above large-amplitude coherent wave groups (freak events) rather than individual steep waves.

The estimator, which is actually a kind of an autobicorrelation function, turns out to be efficient to reveal wave coherence in all cases, when the self-modulation effects are significant. The deep-water frequency spectrum is twice narrower than the wavenumber spectrum, and, thus, probably even a stronger wave coherence might be observed in the frequency spectrum. Indeed, a preliminary analysis corroborated the ability of the suggested autobicorrelation estimator to reveal the coherence in laboratory-measured time series of the surface displacement.

We conclude that in the case of modulationally unstable wave fields the wave correlation is quite significant and can be revealed in rather close vicinity to the spectral peak. This part of the spectrum cannot be considered as a superposition of independent waves with random phases. The phase coherence should be taken into account when defining the nonlinear waves accurately.

REFERENCES

[1] N. Akhmediev, A. Ankiewicz, M. Taki. Waves that appear from nowhere and disappear without a trace. *Phys. Lett. A*, 373:675-678, 2009.

[2] I.E. Alber. The effects of randomness on the stability of two-dimensional wavetrains. *Proc. Roy. Soc. Lond. A*, 363:525-546, 1978.

[3] S.Yu. Annenkov, V. Shrira. Role of non-resonant interactions in the evolution of nonlinear random water wave fields. *J. Fluid Mech.*, 561:181-207, 2006.

[4] S.Y. Annenkov, V.I. Shrira. Evolution of kurtosis for wind waves. *Geophys. Res. Lett.*, 36:L13603, 2009.

[5] S.Y. Annenkov, V.I. Shrira. "Fast" nonlinear evolution in wave turbulence. *Phys. Rev. Lett.*, 102:024502, 2009.

[6] D. Chalikov. Freak waves: Their occurrence and probability. *Phys. Fluids*, 21:076602, 2009.

[7] D. Clamond, M. Francius, J. Grue, C. Kharif. Long time interaction of envelope solitons and freak wave formations. *Eur. J. Mech. B / Fluids*, 25:536-553, 2006.

[8] A.I. Dyachenko, V.E. Zakharov. Modulation instability of stokes wave $\rightarrow$ freak wave. *JETP Lett.*, 81:255-259, 2005.

[9] A.I. Dyachenko, V.E . Zakharov. On the formation of freak waves on the surface of deep water. *JETP Lett.*, 88:307-310, 2008.

[10] K.B. Dysthe. Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. Roy. Soc. London A*, 369:105-114, 1979.

[11] K. Dysthe, H.E. Krogstad, P. Muller. Oceanic rogue waves. *Annu. Rev. Fluid Mech.*, 40:287-310, 2008.

[12] K.B. Dysthe, K. Trulsen. Note on breather type solutions of the NLS as a model for freak-waves. *Physica Scripta T*, 82:48-52, 1999.

[13] K.B. Dysthe, K. Trulsen, H.E. Krogstad, H. Socquet-Juglard. Evolution of a narrow-band spectrum of random surface gravity waves. *J. Fluid. Mech.*, 478:1-10, 2003.

[14] K.L. Henderson, D.H. Peregrine, J.W. Dold. Unsteady water wave modulations: fully nonlinear solutions and comparison with the nonlinear Schrodinger equation. *Wave Motion*, 29:341-361, 1999.

[15] P.A.E.M. Janssen Nonlinear four-wave interactions and freak waves. *J. Phys. Oceanogr.*, 33:863-884, 2003.

[16] R.S. Johnson. *A modern introduction to the mathematical theory of water waves.* Cambridge Univ. Press, 1997.

[17] E. Kartashova, C. Raab, Ch. Feurer, G. Mayrhofer, W. Schreiner. Symbolic Computation for Nonlinear Wave Resonances. In E. Pelinovsky, C. Kharif, editors, *Extreme Waves.* Springer, 95-126, 2008.

[18] C. Kharif, E. Pelinovsky. Physical mechanisms of the rogue wave phenomenon. *Eur. J. Mech. / B - Fluids*, 22:603-634, 2003.

[19] C. Kharif, E. Pelinovsky, A. Slunyaev. *Rogue Waves in the Ocean.* Springer-Verlag, Berlin Heidelberg, 2009.

[20] N. Mori, M. Onorato, P.A.E.M. Janssen, A.R. Osborne, M. Serio. On the extreme statistics of long-crested deep water waves: Theory and experiments. *J. Geophys. Res.*, 112:C09011, 2007.

[21] M. Onorato, A.R. Osborne, M. Serio, S. Bertone. Freak waves in random oceanic sea states. *Phys. Rev. Lett.*, 86:5831-5834, 2001.

[22] M. Onorato, A.R. Osborne, M. Serio. Extreme wave events in directional, random oceanic sea states. *Phys. Fluids*, 14:L25-L28, 2002.

[23] M. Onorato, A.R. Osborne, M. Serio, L. Cavaleri. Modulational instability and non-Gaussian statistics in experimental random water-wave trains. *Phys. Fluids*, 17:078101, 2005.

[24] M. Onorato, T. Waseda, A. Toffoli, L. Cavaleri, O. Gramstad, P.A. Janssen, T. Kinoshita, J. Monbaliu, N. Mori, A.R. Osborne, M. Serio, C.T. Stansberg, H. Tamura, K.M. Trulsen. Statistical properties of directional ocean waves: the role of the modulational instability in the formation of extreme events. *Phys. Rev. Lett.*, 102:114502, 2009.

[25] A.R. Osborne, M. Onorato, M. Serio. The nonlinear dynamics of rogue waves and holes in deep water gravity wave trains. *Phys. Lett. A*, 275:386-393, 2000.

[26] L. Shemer, A. Sergeeva. An experimental study of spatial evolution of statistical parameters in a unidirectional narrow-banded random wavefield. *J. Geophys. Res.*, 114:C01015, 2009.

[27] L. Shemer, A. Sergeeva, A. Slunyaev. Applicability of envelope model equations for simulation of narrow-spectrum unidirectional random field evolution: experimental validation. *Phys. Fluids*, 22:016601, 2010.

[28] L. Shemer, A. Sergeeva, D. Liberzon. Effect of the initial spectral shape on spatial evolution of the statistics of unidirectional nonlinear random waves. *J. Geophys. Res.* (Submitted)

[29] A.V. Slunyaev. Numerical simulation of "limiting" envelope solitons of gravity waves on deep water. *JETP*, 109:676-686, 2009.

[30] A.V. Slunyaev. Freak wave events and the wave phase coherence. *Europ. Phys. J.* (Submitted)

[31] H. Socquet-Juglard, K.B. Dysthe, K. Trulsen, H.E. Krogstad, J. Liu. Probability distributions of surface gravity waves during spectral changes. *J. Fluid Mech.*, 542:195-216, 2005.

[32] M. Stiassnie, L. Shemer. On the interaction of four water-waves. *Wave Motion*, 41:307-328, 2005.

[33] K. Trulsen, I. Kliakhandler, K.B. Dysthe, M.G. Velarde. On weakly nonlinear modulation of waves on deep water. *Phys. Fluids*, 12:2432-2437, 2000.

[34] V.E. Zakharov, L.A. Ostrovsky. Modulation instability: the beginning. *Physica D*, 238:540-548, 2009.