Sequential Fractional Differential Equations with Hadamard Derivative

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Abstract: A class of nonlinear sequential fractional differential equations dependent on the basic fractional operator involving a Hadamard derivative is studied for arbitrary real noninteger order $\alpha \in \mathbb{R}_+$. The existence and uniqueness of the solution is proved using the contraction principle and a new, equivalent norm and metric, introduced in the paper. As an example, a linear nonhomogeneous FDE is solved explicitly in arbitrary interval $[a, b]$ and for a nonhomogeneous term given as an arbitrary Fox function. The general solution consists of the solution of a homogeneous counterpart equation and a particular solution corresponding to the nonhomogeneous term and is given as a linear combination of the respective Fox functions series.

Keywords: Fractional differential equation, Hadamard derivative, Banach theorem, existence and uniqueness of solutions, equivalent metrics, Fox functions series

1. INTRODUCTION

During the last decades, fractional differential equations (FDE) have become an important tool in the mathematical modelling of many systems and processes in mechanics, physics, chemistry, economics, engineering and bioengineering. The investigations concerning solving methods, both analytical and numerical, the existence and uniqueness of solutions, as well as studies of the properties of solutions have yielded many important results and made FDE theory an important part of applied and pure mathematics (compare monographs and review papers (1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11) and the references given therein).

In the paper we study a class of nonlinear sequential fractional differential equations (SFDE) dependent on the basic $\left(\log \frac{t}{a}\right)^{\beta} \mathcal{D}^{\alpha}_{a+}$ - operator with the Hadamard derivative of arbitrary real noninteger order $\alpha \in \mathbb{R}_+$ and real parameter $\beta \in \mathbb{R}$. Let us note that the Cauchy problem for FDE containing the Hadamard derivatives was discussed in (4) in a non-sequential setting. On the other hand, results on the existence and uniqueness of solutions for SFDE with a basic Riemann-Liouville derivative can be found in (4; 12; 13; 14).

To prove the existence and uniqueness of the solutions in an arbitrary finite interval $[a, b]$ we follow the fixed point method and apply the Banach theorem. A crucial point in the proof is the application of a newly-introduced class of one-parameter equivalent norms (and respective metrics) in the space of continuous weighted functions. In this space the SFDE in the form of

$$\left[\left(\left(\log \frac{t}{a}\right)^{\beta} \mathcal{D}^{\alpha}_{a+}\right)^{m} - \mathbf{L}\left(\left(\log \frac{t}{a}\right)^{\beta} \mathcal{D}^{\alpha}_{a+}\right)\right] f(t) =$$

$$= \Psi(t, f(t)),$$

where $\Psi \in C([a, b] \times \mathbb{R})$ and

$$\mathbf{L}(\lambda) := \sum_{j=0}^{m-1} c_j \lambda^j, \tag{1}$$

is equivalent to the following fractional integral equation

$$f(t) = \bar{\mathbf{L}}\left(\mathcal{I}^{\alpha}_{a+}\left(\log \frac{t}{a}\right)^{-\beta}\right) f(t) +$$

$$+ \left(\mathcal{I}^{\alpha}_{a+}\left(\log \frac{t}{a}\right)^{-\beta}\right)^{m} \Psi(t, f(t)) + \phi_0(t),$$

provided $f \in C_{n-\alpha,\log}[a, b]$ the function space described in (5), constants $a, b \in \mathbb{R}_+$ arbitrary, function ϕ_0 belongs to the kernel of the $\left(\left(\log \frac{t}{a}\right)^{\beta} \mathcal{D}^{\alpha}_{a+}\right)^{m}$ operator and

$$\bar{\mathbf{L}}(\lambda) := \sum_{j=0}^{m-1} c_j \lambda^{m-j}. \tag{2}$$

We shall prove that the integral operator on the right-hand side of the above equation yields a contractive mapping in the space of continuous weighted functions $C_{n-\alpha,\log}[a, b]$, when it is endowed with a respective metric from the introduced family of one-parameter metrics.

As an application, we shall study in detail a simple linear nonhomoegeneous FDE in the case when the nonhomogeneous term is chosen from the family of Fox functions. Using the integration properties of Fox functions we arrive at an explicit form of the general solution given as a Fox functions series. The convergence of the respective series results from the main theorem.

The paper is organized as follows. In the next section we recall all the necessary definitions and properties of fractional operators and Fox functions. We also construct a one-parameter class of equivalent norms and respective metrics in the space of continuous weighted functions $C_{n-\alpha,\log}[a,b]$. Then we prove that certain fractional integral operators are bounded in this space endowed with an arbitrary norm from the proposed class. We generalize the basic integral operator to a mapping, which appears to be a contraction under the respective assumptions on a parameter determining the norm and metric on the function space. Section 3 contains the main result - a theorem on the existence and uniqueness of the solution to a certain nonlinear sequential FDE with the Hadamard derivative. In section 4, applications to the linear and nonhomogeneous FDE are given. For the arbitrary order of a fractional derivative, we derive an explicit general solution in the form of a Fox function series.

2. PRELIMINARIES

In the paper we shall study the existence and explicit form of solutions of a certain fractional differential equation in the $C_{\gamma,\log}[a,b]$ space when $\gamma = n-\alpha$. Let us recall the norm $||\cdot||_{\gamma,\log}$ and the generated metric active in this space of weighted continuous functions, when $Re(\gamma) \in (0,1)$:

$$|| f ||_{\gamma,\log} := \sup_{t\in[a,b]} | \left(\log\frac{t}{a}\right)^{\gamma} f(t) | \tag{3}$$

$$d(f,g) := || f-g ||_{\gamma,\log} . \tag{4}$$

The $C_\gamma[a,b]$ space is then given as

$$C_{\gamma,\log}[a,b] := \{f \in C(a,b]; \;\; || f ||_{\gamma,\log} < \infty\}. \tag{5}$$

Now, we recall the definitions of left-sided fractional operators. In our paper we shall consider fractional differential equations containing Hadamard derivatives. Both the integral and derivative are defined as follows (4).

Definition 2.1 *Let $Re(\alpha) > 0$. Then the left-sided Hadamard integral of order α is given by the formula*

$$(\mathcal{I}^{\alpha}_{a+}f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds \quad t > a > 0, \tag{6}$$

where $\Gamma(\alpha)$ denotes the Euler gamma function.
Let $Re(\alpha) \in (n-1,n)$. Then the left-sided Hadamard derivative is defined as

$$(\mathcal{D}^{\alpha}_{a+}f)(t) = \left(t\frac{d}{dt}\right)^{n}(\mathcal{I}^{n-\alpha}_{a+}f)(t) \qquad t > a > 0. \tag{7}$$

An important and characteristic feature of the above fractional operators is their composition rule which we quote in property below after the monograph by Kilbas et al (4). It will be applied in the transformation of the investigated FDE into its equivalent integral form as well as in the derivation of the corresponding initial conditions.

Property 2.2 *Let $Re(\beta) \geq Re(\alpha) > 0$. Then the following formula*

$$\mathcal{D}^{\alpha}_{a+}\mathcal{I}^{\beta}_{a+}f(t) = \mathcal{I}^{\beta-\alpha}_{a+}f(t) \tag{8}$$

holds at any point $t \in [a,b]$ when $f \in C[a,b]$. If $f \in C_{\gamma,\log}[a,b]$, then the above composition rule holds at any point $t \in (a,b]$.

Let us also recall some results on Hadamard integration and differentiation which are useful in the derivation of stationary functions and in the construction of solutions to the discussed equations.

Property 2.3 *Let $Re(\beta) > 0$ and $Re(\alpha) > 0$. Then the following formulas hold:*

$$\mathcal{I}^{\alpha}_{a+}\left(\log\frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1} \tag{9}$$

$$\mathcal{D}^{\alpha}_{a+}\left(\log\frac{t}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1} . \tag{10}$$

In further considerations we also apply Fox functions. Such functions are defined via a Mellin-Barnes integral and look as follows for integer numbers m,n,p,q fulfilling $0 \leq m \leq q,\; 0 \leq n \leq p$, for complex numbers $a_i, b_j \in \mathbb{C}$ and for real parameters $\alpha_i, \beta_j \in \mathbb{R}$ (4; 16):

$$H^{m,n}_{p,q}\left[z \left| \begin{matrix}(a_i)_{1,p};(\alpha_i)_{1,p}\\(b_j)_{1,q};(\beta_j)_{1,q}\end{matrix}\right.\right] = \tag{11}$$

$$\frac{1}{2\pi i}\int_{\mathfrak{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)} z^{-s}ds,$$

where $i = 1,2,\ldots,p; j = 1,2,\ldots,q$ and contour $\mathfrak{L}$ separates the poles of the gamma functions in the numerator of the complex kernel (4; 16).

When $\alpha_i = 1, \beta_j = 1$ for any $i = 1,\ldots,p$ $j = 1,\ldots,q$, the above Fox function belongs to the subclass of Meijer G-functions:

$$H^{m,n}_{p,q}\left[z \left| \begin{matrix}(a_i)_{1,p};(1)_{1,p}\\(b_j)_{1,q};(1)_{1,q}\end{matrix}\right.\right] = G^{m,n}_{p,q}\left[z \left| \begin{matrix}(a_i)_{1,p}\\(b_j)_{1,q}\end{matrix}\right.\right]. \tag{12}$$

The following parameters determine the integration properties of Fox and Meijer functions:

$$a^* := \sum_{i=1}^{n}\alpha_i - \sum_{i=n+1}^{p}\alpha_i + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j \tag{13}$$

$$\mu := \sum_{j=1}^{q}b_j - \sum_{i=1}^{p}a_i + \frac{p-q}{2} \tag{14}$$

In what follows, we shall study a class of equations containing the fractional differential operators of a real order. Thus, we assume order $\alpha \in \mathbb{R}_+$ as well as parameter $\beta \in \mathbb{R}$ throughout the paper. In the property below, we rewrite the general theorem on the Riemann-Liouville integration of Fox functions to the case of real α and β (compare the definition of the I^{α}_{0+} integral and Theorem 2.7 from monograph (16)).

Property 2.4 *Let us assume $H^{m,n}_{p,q} \in C_{n-\alpha}[0,b]$ and $a^* > 0$ or $a^* = 0, Re(\mu) < -1$. If $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}$ and $\{\alpha\} - \beta > 0$, then the following integration formula holds:*

$$I^{\alpha}_{0+}t^{-\beta}H^{m,n}_{p,q}\left[t^{\sigma} \left| \begin{matrix}(a_i)_{1,p};(\alpha_i)_{1,p}\\(b_j)_{1,q};(\beta_j)_{1,q}\end{matrix}\right.\right] = \tag{15}$$

$$= t^{\alpha-\beta} H^{m,n+1}_{p+1,q+1} \left[t^{\sigma} \left| \begin{matrix} (\beta,(a_i)_{1,p});(\sigma,(\alpha_i)_{1,p}) \\ ((b_j)_{1,q},\beta-\alpha);((\beta_j)_{1,q},\sigma) \end{matrix} \right. \right].$$

In the above property the results are given for the Riemann-Liouville fractional integral. This formula can be translated for the Hadamard integral when we apply the following relation between both integrals:

$$N_{log} f(t) := f(\log \frac{t}{a}) \tag{16}$$

$$N_{log} I^{\alpha}_{0+} N^{-1}_{log} = \mathcal{I}^{\alpha}_{a+}. \tag{17}$$

Using the introduced relation we arrive at an analogous integration formula which describes the Hadamard integral for the respective Fox function.

Property 2.5 *Let us assume $H^{m,n}_{p,q} \in C_{n-\alpha}[0,b]$ and $a^* > 0$ or $a^* = 0, Re(\mu) < -1$. If $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}$ and $\{\alpha\} - \beta > 0$, then the following integration formula holds:*

$$\mathcal{I}^{\alpha}_{a+} N_{log} t^{-\beta} H^{m,n}_{p,q} \left[t^{\sigma} \left| \begin{matrix} (a_i)_{1,p};(\alpha_i)_{1,p} \\ (b_j)_{1,q};(\beta_j)_{1,q} \end{matrix} \right. \right] = \tag{18}$$

$$= N_{\log} t^{\alpha-\beta} H^{m,n+1}_{p+1,q+1} \left[t^{\sigma} \left| \begin{matrix} (\beta,(a_i)_{1,p});(\sigma,(\alpha_i)_{1,p}) \\ ((b_j)_{1,q},\beta-\alpha);((\beta_j)_{1,q},\sigma) \end{matrix} \right. \right].$$

To solve the discussed equation in the space of continuous weighted functions, we shall extend the standard norm and metric (3,4). To this aim, we propose to apply a two-parameter composed Mittag-Leffler function (compare monograph (4)). Let us define a three-parameter family of functions using such a Mittag-Leffler function. For real numbers $\alpha, \beta, \kappa \in \mathbb{R}$ we define function $e^{\alpha,\beta,\kappa}$ as follows:

$$e^{\alpha,\beta,\kappa}(t) := \Gamma(\{\alpha\}-\beta) E_{\alpha,\{\alpha\}-\beta} \left(\kappa \left(\log \frac{t}{a} \right)^{\alpha} \right), \tag{19}$$

where E denotes the two-parameter Mittag-Leffler function given in our construction as series

$$E_{\alpha,\{\alpha\}-\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \{\alpha\} - \beta)}, \tag{20}$$

where the $z \in \mathbb{C}$ arbitrary and $\{\alpha\}$ denotes the fractional part of real number α. It is easy to check that functions $e^{\alpha,\beta,\kappa}$ obey the following fractional integration formula:

$$\mathcal{I}^{\alpha}_{a+} \left(\log \frac{t}{a} \right)^{\{\alpha\}-\beta-1} e^{\alpha,\beta,\kappa}(t) = \tag{21}$$

$$= \frac{\left(\log \frac{t}{a}\right)^{\{\alpha\}-\beta-1}}{\kappa} \left[e^{\alpha,\beta,\kappa}(t) - 1 \right],$$

provided $\{\alpha\} - \beta > 0$.
Let us observe that we can apply the functions defined in (19) to modify norm (3) and the respective metric in the sense of Bielecki (17). He used exponential functions to introduce an equivalent metric and to show the existence of global solutions of certain ordinary and partial differential equations in respective function spaces. A similar technique was proposed in (18; 19) for some simple nonlinear fractional differential equations. Then Lakshmikantham et al (1; 20) developed a modification of the metric in the $C[0,b]$ space by means of a one-parameter Mittag-Leffler function and applied it in the solution of the nonlinear fractional differential equation of order $\alpha \in (0,1)$. Here, we shall construct a class of metrics in the $C_{n-\alpha,\log}[a,b]$ space which are equivalent to standard metric (4) generated by norm (3). These metrics will be applied in the proof of the existence and uniqueness of solutions for a class of fractional differential equations in the $C_{n-\alpha,\log}[a,b]$ space.
Definition 2.6 *The following formulas define a norm and a respective metric in the $C_{n-\alpha,\log}[a,b]$ space, provided $\kappa \in \mathbb{R}_+$ and $\{\alpha\} - \beta > 0$*

$$\| f \|_{n-\alpha,\log,\kappa} := \sup_{t\in[a,b]} \frac{\left| \left(\log \frac{t}{a}\right)^{n-\alpha} f(t) \right|}{e^{\alpha,\beta,\kappa}(t)} \tag{22}$$

$$d_{\kappa}(f,g) := \| f - g \|_{n-\alpha,\log,\kappa}\ . \tag{23}$$

Property 2.7 *Metric d_κ is equivalent to standard metric (4) generated by norm (3).*
Proof: The equivalence of the metrics results from the following inequalities fulfilled by norms (3) and (22):

$$\frac{\| f \|_{n-\alpha,\log}}{e^{\alpha,\beta,\kappa}(b)} \leq \| f \|_{n-\alpha,\log,\kappa} \leq \| f \|_{n-\alpha,\log} \tag{24}$$

for arbitrary function $f \in C_{n-\alpha,\log}[a,b]$.
Let us note that fractional integral operator $\mathcal{I}^{\alpha}_{a+} \left(\log \frac{t}{a}\right)^{-\beta}$ is bounded in the $(C_{n-\alpha,\log}[a,b]; \ \| \cdot \|_{n-\alpha,\log,\kappa})$ spaces when $\beta \leq 0$ and $a, b \in \mathbb{R}_+$ arbitrary. This property follows from formula (21) and from the fact that series $e^{\alpha,\beta,\kappa}$ is absolutely and uniformly convergent in any interval $[a,b]$ when $a > 0$ and assumptions from Definition 2.6 are fulfilled.

Property 2.8 *If $\kappa \in \mathbb{R}_+$ and $\beta \leq 0$, then the following inequalities are valid for all functions $f \in C_{n-\alpha,\log}[a,b]$, $a > 0$:*

$$\| \mathcal{I}^{\alpha}_{a+} \left(\log \frac{t}{a} \right)^{-\beta} f \|_{n-\alpha,\log,\kappa} \leq \tag{25}$$

$$\leq \frac{\left(\log \frac{b}{a}\right)^{-\beta}}{\kappa} \| f \|_{n-\alpha,\log,\kappa}$$

$$\| \left(\mathcal{I}^{\alpha}_{0+} \left(\log \frac{t}{a} \right)^{-\beta} \right)^{j} f \|_{n-\alpha,\log,\kappa} \leq \tag{26}$$

$$\leq \left(\frac{\left(\log \frac{b}{a}\right)^{-\beta}}{\kappa} \right)^{j} \| f \|_{n-\alpha,\log,\kappa}\ .$$

Now we construct a mapping on the space of weighted continuous functions $C_{n-\alpha,\log}[a,b]$ and test its properties.
Definition 2.9 *We define mapping T_m as follows*

$$T_m f(t) := \bar{\mathbf{L}} \left(\mathcal{I}^{\alpha}_{a+} \left(\log \frac{t}{a} \right)^{-\beta} \right) f(t) \tag{27}$$

for any function $f \in C_{n-\alpha,\log}[a,b]$, $\{\alpha\} - \beta \geq 0$ and polynomial $\bar{\mathbf{L}}$ defined in formula (2).
The defined mapping is a contraction in the $(C_{n-\alpha,\log}[a,b]; \ d_\kappa)$ space of functions when the metric is given by formula (23) and parameter κ is large enough. This property is formulated and proved in the following lemma.

Lemma 2.10 *If* $\kappa > (1+\sum_{j=0}^{m-1} \mid c_j \mid) \left(\log \frac{b}{a}\right)^{-\beta}$ *and* $\beta \leq 0$, *then mapping* T_m *given in Definition 2.9 is a contraction in the* $C_{n-\alpha,\log}[a,b]$ *space endowed with the* d_κ *metric.*
Proof: Let f, g be a pair of arbitrary functions from the $C_{n-\alpha,\log}[a,b]$ space. We apply Property 2.8 to estimate the d_κ distance of their images $T_m f$ and $T_m g$:

$$
\begin{aligned}
& d_\kappa(T_m f, T_m g) = || T_m f - T_m g ||_{n-\alpha,\log,\kappa} \leq \\
& \leq \sum_{j=0}^{m-1} \mid c_j \mid \times \\
& \times || \left(\mathcal{I}^\alpha_{0+}\left(\log\frac{t}{a}\right)^{-\beta}\right)^{m-j} (f(t)-g(t)) ||_{n-\alpha,\log,\kappa} \leq \\
& \leq \left(\sum_{j=0}^{m-1} \mid c_j \mid \frac{\left(\log\frac{b}{a}\right)^{-\beta(m-j)}}{\kappa^{m-j}}\right) \cdot || f-g ||_{n-\alpha,\log,\kappa} \leq \\
& \leq \frac{\sum_{j=0}^{m-1} \mid c_j \mid}{\left(\log\frac{b}{a}\right)^{\beta}\kappa} \cdot || f-g ||_{n-\alpha,\log,\kappa} = \\
& = \frac{\sum_{j=0}^{m-1} \mid c_j \mid}{\left(\log\frac{b}{a}\right)^{\beta}\kappa} d_\kappa(f,g).
\end{aligned}
$$

Thus, we obtained inequality

$$
d_\kappa(T_m f, T_m g) \leq \frac{\sum_{j=0}^{m-1} \mid c_j \mid}{\left(\log\frac{b}{a}\right)^{\beta}\kappa} d_\kappa(f,g), \tag{28}
$$

which means T_m is a contractive mapping with respect to the d_κ metric as fraction $(\sum_{j=0}^{m-1} \mid c_j \mid)/\left(\log\frac{b}{a}\right)^{\beta}\kappa \in (0,1)$ by assumption. This ends the proof.
Remark: Let us note that the proved result does not predict what happens when $\{\alpha\}-\beta>0$ and $\beta>0$. In this case norm (3) must be modified using another type of function. Detailed calculations will be given in a subsequent paper.

3. MAIN RESULTS

We shall study the existence and uniqueness of solutions for a sequential fractional differential equation with the Hadamard derivative:

$$
\left[\left(\left(\log\frac{t}{a}\right)^{\beta}\mathcal{D}^\alpha_{a+}\right)^m - \mathbf{L}\left(\left(\log\frac{t}{a}\right)^{\beta}\mathcal{D}^\alpha_{a+}\right)\right] f(t) = \tag{29}
$$
$$
= \Psi(t, f(t)),
$$

when $\{\alpha\}-\beta>0$, $\mathbf{L}$ is polynomial given in (1), $\Psi \in C([a,b]\times\mathbb{R})$ and $t\in[a,b]$.
From the composition rules given in Property 2.2, it follows that the above equation is equivalent in the $C_{n-\alpha,\log}[a,b]$ space to the fractional integral equation written using defined mapping T_m (Definition 2.9):

$$
\begin{aligned}
& f(t) = \\
& = T_m f(t) + \left(\mathcal{I}^\alpha_{a+}\left(\log\frac{t}{a}\right)^{-\beta}\right)^m \Psi(t,f(t)) + \phi_0(t).
\end{aligned} \tag{30}
$$

Let us denote the mapping on the right-hand side of the above equation as follows:

$$
\begin{aligned}
& Tf(t) := \\
& = T_m f(t) + \left(\mathcal{I}^\alpha_{a+}\left(\log\frac{t}{a}\right)^{-\beta}\right)^m \Psi(t,f(t)) + \phi_0(t).
\end{aligned} \tag{31}
$$

Function ϕ_0 in definition (31) is an arbitrary stationary function of operator $((\log\frac{t}{a})^\beta\mathcal{D}^\alpha_{a+})^m$, which means it fulfills the equation

$$
((\log\frac{t}{a})^\beta\mathcal{D}^\alpha_{a+})^m\phi_0(t) = 0. \tag{32}
$$

Such an equation is easy to solve and the solution is a linear combination of the power functions

$$
\phi_0(t) = \sum_{j=0}^{m-1}\sum_{l=1}^{n} d_{j,l}\left(\log\frac{t}{a}\right)^{(\alpha-\beta)j+\alpha-l}, \tag{33}
$$

where coefficients $d_{j,l}$ are arbitrary real numbers. It can also be expressed as a combination of composed Meijer G-functions:

$$
\begin{aligned}
& \phi_0(t) = \\
& = \sum_{j=0}^{m-1}\sum_{l=1}^{n} d'_{j,l} N_{log} G^{1,0}_{1,1}\left[t\left|\begin{matrix}(\alpha-\beta)j+\alpha-l+1\\(\alpha-\beta)j+\alpha-l\end{matrix}\right.\right].
\end{aligned} \tag{34}
$$

Let us observe that due to assumption $\{\alpha\}-\beta>0$, all the stationary functions given by (33,34) belong to the $C_{n-\alpha,\log}[a,b]$ space. Thus, mapping T given in (31), maps the weighted continuous functions into functions from the same space for arbitrary function ϕ_0.

Proposition 3.1 *Let* $\alpha\in(n-1,n)$, $\beta\leq 0$ *and function* $\Psi\in C([a,b]\times\mathbb{R})$ *fulfill the following Lipschitz condition*

$$
\mid \Psi(t,x)-\Psi(t,y) \mid < M \mid x-y \mid \tag{35}
$$

for $t\in[a,b]$ *and* $x,y\in\mathbb{R}$.
Each stationary function ϕ_0 *given by (33,34) generates unique solution* $f\in C_{n-\alpha,\log}[a,b]$ *of fractional differential equation (29).*
The solution is given as a limit of iterations of mapping T *(31):*

$$
f(t) = \lim_{k\longrightarrow\infty} T^k\psi(t), \tag{36}
$$

where function $\psi\in C_{n-\alpha,\log}[a,b]$ *is arbitrary.*
Proof: We start by observing that Lipschitz condition (35) yields the following inequality for each $\kappa\in\mathbb{R}_+$:

$$
\begin{aligned}
& || \Psi(t,f(t))-\Psi(t,g(t)) ||_{n-\alpha,\log,\kappa} = \\
& = \sup_{t\in[a,b]} \frac{(\log\frac{t}{a})^{n-\alpha} \mid \Psi(t,f(t))-\Psi(t,g(t)) \mid}{e^{\alpha,\beta,\kappa}(t)} \leq \\
& \leq M \sup_{t\in[a,b]} \frac{(\log\frac{t}{a})^{n-\alpha} \mid f(t)-g(t) \mid}{e^{\alpha,\beta,\kappa}(t)} = \\
& = M\cdot || f-g ||_{n-\alpha,\log,\kappa}.
\end{aligned}
$$

Condition $\beta\leq 0$ implies $\{\alpha\}-\beta>0$. Due to this fact and to the above calculations, mapping T defined in (31) is bounded for any stationary function ϕ_0 and in each space $(C_{n-\alpha,\log}[a,b];\ ||\cdot||_{n-\alpha,\log,\kappa})$ according to Property 2.8. Now we assume:

$$\kappa > (1+M+\sum_{j=0}^{m-1} \mid c_j \mid)(\log\frac{b}{a})^{-\beta} \tag{37}$$

and write equation (29) as an equivalent fixed point condition

$$f(t) = Tf(t) \tag{38}$$

in space $(C_{n-\alpha,\log}[a,b];\ ||\cdot||_{n-\alpha,\log,\kappa})$. We check that T is a contraction by a straightforward calculation:

$$\begin{aligned}
& d_\kappa(Tf,Tg) = || Tf - Tg ||_{n-\alpha,\log,\kappa} \leq \\
& \leq || T_m f - T_m g ||_{n-\alpha,\log,\kappa} + \\
& + || (\mathcal{I}^\alpha_{a+}(\log\frac{t}{a})^{-\beta})^m [\Psi(t,f(t)) - \Psi(t,g(t))] ||_{n-\alpha,\log,\kappa} \leq \\
& \leq \frac{\sum_{j=0}^{m-1} \mid c_j \mid}{(\log\frac{b}{a})^\beta \kappa} || f-g ||_{n-\alpha,\log,\kappa} + \\
& + \frac{M}{((\log\frac{b}{a})^\beta \kappa)^m} || f-g ||_{n-\alpha,\log,\kappa} \leq \\
& \leq \frac{M + \sum_{j=0}^{m-1} \mid c_j \mid}{(\log\frac{b}{a})^\beta \kappa} || f-g ||_{n-\alpha,\log,\kappa} = \\
& = \frac{M + \sum_{j=0}^{m-1} \mid c_j \mid}{(\log\frac{b}{a})^\beta \kappa} d_\kappa(f,g).
\end{aligned}$$

We conclude that mapping T obeys for any pair of functions $f,g \in C_{n-\alpha,\log}[a,b]$ the following condition:

$$d_\kappa(Tf,Tg) < \frac{M + \sum_{j=0}^{m-1} \mid c_j \mid}{(\log\frac{b}{a})^\beta \kappa} d_\kappa(f,g) \tag{39}$$

which means it is a contraction as fraction $\frac{M+\sum_{j=0}^{m-1}|c_j|}{(\log\frac{b}{a})^\beta\kappa} \in (0,1)$ by assumption (37).

Hence, a unique fixed point in space $C_{n-\alpha,\log}[a,b]$ exists by the Banach theorem and is explicitly given as a limit of iterations of mapping T:

$$f(t) = \lim_{k\longrightarrow\infty} T^k \psi(t), \tag{40}$$

where $\psi \in C_{n-\alpha,\log}[a,b]$ is an arbitrary starting function and the convergence with respect to the d_κ metric is equivalent to the convergence with respect to standard d metric (4). This ends the proof.

By carefully analysing the above proposition, we notice a one to one correspondence between the choice of the generating stationary function and the unique solution of problem (29). Thus, the initial conditions at $t=a$ are also determined by function ϕ_0. This relation leads to the following formulation and solution of the respective Cauchy problem in the $C_{n-\alpha,\log}[a,b]$ space.

Theorem 3.2 *Let $\alpha \in (n-1,n)$, $\beta \leq 0$ and let function Ψ obey the assumptions of Proposition 3.1. Then fractional differential equation (29) has a unique solution f in the $C_{n-\alpha,\log}[a,b]$ space fulfilling the set of initial conditions:*

$$\mathcal{D}^{\alpha-l}_{a+}((\log\frac{t}{a})^\beta \mathcal{D}^\alpha_{a+})^j f(t) \mid_{t=a} = \bar{d}_{j,l}, \tag{41}$$

where $l=1,\ldots,n$ and $j=0,\ldots,m-1$.
This solution is a limit of the iterations of mapping T (31) generated by a stationary function in the form of

$$\phi_0(t) = \tag{42}$$

$$= \sum_{j=0}^{m-1}\sum_{l=1}^{n} \frac{\bar{d}_{j,l}}{\Gamma((\alpha-\beta)j+\alpha-l+1)} (\log\frac{t}{a})^{(\alpha-\beta)j+\alpha-l}.$$

4. APPLICATIONS - EXAMPLE

We shall now discuss the application of the above results to a nonhomogeneous linear fractional differental equation with the Hadamard derivative:

$$\left[(\log\frac{t}{a})^\beta \mathcal{D}^\alpha_{a+} - \lambda\right] f(t) = N_{log} H^{m,n}_{p,q}(t^\sigma), \tag{43}$$

where $t \in [a,b]$, function H is an arbitrary Fox function from the $C_{n-\alpha}[0,b]$ class, fulfilling the assumptions of Property 2.5 and $\sigma > 0$ (compare formula (11)):

$$H^{m,n}_{p,q}(t^\sigma) = H^{m,n}_{p,q}\left[t^\sigma \left| \begin{matrix} (a_i)_{1,p};(\alpha_i)_{1,p} \\ (b_j)_{1,q};(\beta_j)_{1,q} \end{matrix} \right.\right]. \tag{44}$$

Let us denote the vectors defining the above Fox function as follows:

$$\boldsymbol{a} = (a_i)_{1,p} \quad \boldsymbol{\alpha} = (\alpha_i)_{1,p} \tag{45}$$

$$\boldsymbol{b} = (b_j)_{1,q} \quad \boldsymbol{\beta} = (\beta_j)_{1,q}. \tag{46}$$

The solution of equation (43) is constructed using the integration formula from Property 2.5. The convergence of the respective series is implied and ensured by Proposition 3.1. In this paper we shall restrict the example to a general solution to problem (43), dependent on n constant coefficients.

As we are discussing a linear equation, we know that using the iterations of mappings T, generated by the corresponding stationary functions, we can split the solution into f_0, which solves the homogenous counterpart of equation (43):

$$\left[(\log\frac{t}{a})^\beta \mathcal{D}^\alpha_{a+} - \lambda\right] f_0(t) = 0 \tag{47}$$

and f^s which is generated by stationary function $\phi_0 = 0$. The full solution of equation (43) is then the following sum:

$$f(t) = f_0(t) + f^s(t). \tag{48}$$

Part f^s of the full solution is generated by the $\phi_0 = 0$ stationary function when we also take $\psi = 0$ in (31,40). In this case, mapping T is given for any function $g \in C_{n-\alpha,\log}[a,b]$ by the formula:

$$Tg := \lambda \mathcal{I}^\alpha_{a+}(\log\frac{t}{a})^{-\beta} g + \mathcal{I}^\alpha_{a+}(\log\frac{t}{a})^{-\beta} N_{\log} H^{m,n}_{p,q}(t^\sigma). \tag{49}$$

This implies that f^s is given as a series:

$$f^s(t) = \sum_{k=0}^{\infty} \lambda^k (\mathcal{I}^\alpha_{a+}(\log\frac{t}{a})^{-\beta})^{k+1} N_{\log} H^{m,n}_{p,q}(t^\sigma) \tag{50}$$

which is absolutely convergent with respect to the $||\cdot||_{n-\alpha,\log}$ norm. Function f^s can be explicitly calculated using integration formula (18) and it is the following series of Fox functions:

$$f^s(t) = \sum_{k=0}^{\infty} \lambda^k (\log \frac{t}{a})^{(k+1)(\alpha-\beta)} \times \tag{51}$$

$$\times N_{\log} H^{m,n+k+1}_{p+k+1,q+k+1} N_{log} \left[t^{\sigma} \middle| \begin{matrix} \mathbf{E}_{k+1}; \mathcal{E}_{k+1} \\ \mathbf{F}_{k+1}; \mathcal{F}_{k+1} \end{matrix} \right],$$

where the component vectors look as follows for $k' \geq 1$:

$$\mathbf{E}_{k'} = [\beta \mathbf{e}_{k'} + (\beta - \alpha)\mathbf{j}_{k'}; \mathbf{a}] \in \mathbb{R}^{k'+p} \tag{52}$$

$$\mathbf{F}_{k'} = [\mathbf{b}; (\beta - \alpha)(\mathbf{e}_{k'} + \mathbf{j}_{k'})] \in \mathbb{R}^{k'+q} \tag{53}$$

$$\mathcal{E}_{k'} = [\sigma \mathbf{e}_{k'}; \boldsymbol{\alpha}] \in \mathbb{R}^{k'+p} \tag{54}$$

$$\mathcal{F}_{k'} = [\boldsymbol{\beta}; \sigma \mathbf{e}_{k'}] \in \mathbb{R}^{k'+q}. \tag{55}$$

Vectors $\boldsymbol{a}, \boldsymbol{\alpha}, \boldsymbol{b}, \boldsymbol{\beta}$ are given in (45,46) and we have denoted:

$$\mathbf{e}_{k'} = [1, \ldots, 1] \in \mathbb{R}^{k'} \tag{56}$$

$$\mathbf{j}_{k'} = [0, 1, \ldots, k' - 1] \in \mathbb{R}^{k'}. \tag{57}$$

The f_0 part of the solution is a linear combination of basic solutions:

$$f_0(t) = \sum_{l=1}^{n} a_l f^l(t). \tag{58}$$

Each of the f^l basic components of the solution is generated by the $(\log \frac{t}{a})^{\alpha-l}$ component of the stationary function and is given as the following series of Meijer G-functions:

$$f^l(t) = \sum_{k=0}^{\infty} \lambda^k (\log \frac{t}{a})^{(\alpha-\beta)k} N_{\log} G^{1,k}_{k+1,k+1} \left[t \middle| \begin{matrix} \mathbf{A}_{k,l} \\ \mathbf{B}_{k,l} \end{matrix} \right] \tag{59}$$

with defining vectors $\mathbf{A}_{k,l}, \mathbf{B}_{k,l}$

$$\mathbf{A}_{k,l} = [\beta \mathbf{e}_k + (\beta - \alpha)\mathbf{j}_k; \alpha - l + 1] \in \mathbb{R}^{k+1} \tag{60}$$

$$\mathbf{B}_{k,l} = [\alpha - l; (\beta - \alpha)(\mathbf{e}_k + \mathbf{j}_k)] \in \mathbb{R}^{k+1}. \tag{61}$$

In the above formulas $\mathbf{e}_k, \mathbf{j}_k$ are given in (56,57).

5. FINAL REMARKS

In the paper we derived an explicit solution for a class of SFDE with the Hadamard derivative. It is given as a limit of contractions in the space of continuous weighted functions with a respective metric. As all the newly constructed metrics are equivalent to the standard metric, we observe that the proved convergence also provides the convergence with respect to the initial metric.

The case of a simple linear nonhomogeneous fractional equation is discussed in detail and the solutions are Fox functions series. This form of the solution appears to be similar to that obtained earlier for FDE (11).

Let us note that the proposed method of proving the existence and uniqueness of a solution to a SFDE can be easily extended to the case when a fractional operator of another type is involved. Our result, valid in the space of continuous weighted functions, and that obtained in (1; 18; 19; 20) for continuous functions, imply that the extension of the Bielecki method (17) to FDE theory requires careful construction of the equivalent norms. The respective function space, the form of the basic fractional operator and the properties of Mittag -Leffler functions determine new norms which allow us to prove the global existence of a solution in an arbitrary finite interval.

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