Formulation of A Space-Time Fractional Anomalous Diffusion Problem in Two Dimensional Space

Necati Özdemir * **Derya Avcı** **

* *Department of Mathematics, Faculty of Science and Arts, Balıkesir University, Balıkesir, TURKEY (e-mail:nozdemir@ balikesir.edu.tr)*
** *e-mail:dkaradeniz@balikesir.edu.tr*

Abstract: In this paper, we present an initial value problem for a space-time fractional anomalous diffusion process in two dimensional space. From the mathematical and physical point of views, anomalous diffusion may be based on generalized diffusion equations which are defined in terms of fractional order derivatives in space and/or time. Space derivatives with respect to x and y variables are defined in terms of Riesz-Feller derivative of order $0 < \alpha < 1$ and $1 < \mu \leq 2$, respectively; θ_1 ($\theta_1 \leq \min\{\alpha, 1-\alpha\}$) and θ_2 ($\theta_2 \leq \min\{\mu, 2-\mu\}$) are skewness parameters; and the time derivative is defined in sense of Caputo of order β ($0 < \beta \leq 1$). It is assumed that the solution and the initial condition functions can be expanded in a complex Fourier series. Under this assumption and by applying Fourier-Laplace transforms, analytical solution is obtained. Note that, the expansion of the solution in a series makes it possible to reduce the main space-time fractional differential equation with three fractional derivative terms to a time fractional differential equation. It gives simplifying to calculation of numerical solutions. Grünwald-Letnikov approximation of Caputo derivative is used to take numerical solutions. After that, the comparison of analytical and numerical solutions is proposed by a numerical example and variations of problem parameters are analyzed by figures. Finally, the convergence of analytical and numerical solutions to each other shows the effectiveness of the numerical methods to the present work.

Keywords: Caputo, Riesz-Feller, Grünwald-Letnikov, Fourier Transform, two dimensional.

1. INTRODUCTION

In the last decade, there has been a considerable interest to the applications of fractional calculus such that many processes in the nature have been successfully modelled by a set of axioms, definitions and methods of fractional calculus (see Kilbas et al. (2006), Miller and Ross (1993), Samko and Kilbas (1993), Podlubny (1999)). One of these processes is anomalous diffusion which is a phenomenon occurs in complex and non-homogeneous mediums. The phenomenon of anomalous diffusion may be based on generalized diffusion equation which contains fractional order space and/or time derivatives Metzler and Klafter (2000). Turski et al. (2007) presented the occurrence of the anomalous diffusion from the physical point of view and also explained the effects of fractional derivatives in space and/or time to diffusion propagation. Agrawal (2001) represented an analytical technique by using eigenfunctions for a fractional diffusion-wave system and therefore, provided that this formulation could be applied to all those systems for which the existence of eigenmodes is guaranteed. Agrawal (2002) also formulated a general solution by using finite sine transform technique for a fractional diffusion-wave equation in a bounded domain whose fractional term was described in sense of Caputo. Herzallah et al. (2010) researched the solution of a fractional diffusion wave model which is more accurate and provides the existence, uniqueness and continuation of the solution. Huang and Liu (2005a) considered a sort of generalized diffusion equation which is defined as a space-time fractional diffusion equation in sense of Caputo and Riemann-Liouville operators. In addition, Huang and Liu (2005b) found the fundamental solution of the space-time fractional advection-dispersion equation with Riesz-Feller derivative. Langlands (2006) proposed a modified fractional diffusion equation on an infinite domain and therefore found the solution as an infinite series of Fox functions. Sokolov et al. (2004) analyzed different types of distributed-order fractional diffusion equations and investigated the effects of different classes of such equations. Saichev and Zaslavsky (1997) presented the solutions of a symmetrized fractional diffusion equation with a source term applying a method similar to separation of variables. Mainardi et al. (2001) researched the fundamental solution of a Cauchy problem for the space-time fractional diffusion equation obtained from the standard diffusion equation by replacing the second order space derivative by a fractional Riesz or Riesz-Feller derivative, and the first order time-derivative by a fractional Caputo derivative. Gorenflo and Mainardi (1998), Gorenflo and Mainardi (1999), analyzed a space-fractional (or Levy-Feller) diffusion process governed by a generalized diffusion equation which generates all Levy stable probability distributions and also approx-

* Corresponding author

imated these processes by random walk models, discreted space and time based on Grünwald-Letnikov (GL) approximation. Özdemir et al. (2009) presented the numerical solution of a diffusion-wave problem in polar coordinates by using GL approximation. Özdemir and Karadeniz (2008) also applied GL formula to find the numerical results for a diffusion problem in cylindrical coordinates.

In addition, numerical schemes are very popular research topics in fractional calculus. Because, the analytical solutions of the fractional differential equations are usually obtained in terms of Green and Fox functions and difficult to calculate explicitly. For this reason, there are many research related with numerical approximation of space or space-time fractional diffusion equations. Shen and Liu (2005) investigated the error analysis of the numerical solution of a space fractional diffusion equation obtained by using an explicit finite difference method. Liu et al. (2007) formulated the numerical solution of a space-time fractional advection-dispersion equation in terms of Caputo and RL derivatives by using an implicit and an explicit difference methods. Lin et al. (2009) considered a nonlinear fractional diffusion equation in terms of generalized Riesz fractional derivative and applied an explicit finite-difference method to find numerical solutions. Ciesielski and Leszczynski (2006) proposed a new numerical method for the spatial derivative called Riesz-Feller operator, and so found the numerical solutions to a fractional partial differential equation which describe an initial-boundary value problem in one-dimensional space. Ciesielski and Leszczynski (2006) also presented the numerical solutions of a boundary value problem for an equation with the Riesz-Feller derivative. Liu et al. (2007) presented a random walk model for approximating a Levy-Feller advection-dispersion process and proposed an explicit finite difference approximation for Levy-Feller advection-dispersion process, resulting from the GL discretization of fractional derivatives. Zhang et al. (2007) considered the Levy-Feller diffusion equation and investigated their probabilistic interpretation and numerical analysis in a bounded spatial domain. Moreover, Machado (2003) presented a probabilistic interpretation to the fractional-order derivatives.

The plan of this work as follows. In this work, we consider a two dimensional anomalous diffusion problem in terms of Caputo and Riesz-Feller derivatives. For this purpose, we give some basic definitions necessary for our formulations in Section 2. In section 3, we formulate our considerations and find the analytical solution of the problem. We apply GL definition to find the numerical solution in Section 4. In Section 5, we choose an example and therefore show the effectiveness of the numerical approximation for our problem. Finally, we conclude our work in Section 6.

2. MATHEMATICAL BACKGROUND

In this work, we consider an anomalous diffusion equation in two-dimensional space. We define our problem in terms of Caputo time and Riesz-Feller fractional derivatives. Therefore, let we remind the well known definitions and origins of these operators.

Originally, Riesz introduced the pseudo-differential operator ${}_xI_0^\alpha$ whose symbol is $|\kappa|^{-\alpha}$, well defined for any positive α with the exclusion of odd integer numbers, then was called Riesz Potential. The Riesz fractional derivative ${}_xD_0^\alpha = -\,{}_xI_0^\alpha$ defined by analytical continuation can be represented as follows

$$\begin{aligned} {}_xD_0^\alpha &= -\left|\kappa\right|^\alpha \\ &= -\left(\kappa^2\right)^{\frac{\alpha}{2}} \\ &= -\left(-\frac{d^2}{dx^2}\right)^{\frac{\alpha}{2}}. \end{aligned} \tag{1}$$

In addition, Feller (1952) generalized the Riesz fractional derivative to include the skewness parameter θ of the strictly stable densities. Feller showed that the pseudo-differential operator D_θ^α is as the inverse to the Feller potential, which is a linear combination of two Riemann-Liouville (or Weyl) integrals:

$${}_xI_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{x} (x-\xi)^{\alpha-1} f(\xi)\, d\xi, \tag{2}$$

$${}_xI_-^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{x}^{+\infty} (\xi-x)^{\alpha-1} f(\xi)\, d\xi \tag{3}$$

where $\alpha > 0$. By these definitions the Feller potential can be defined as

$${}_xI_\theta^\alpha f(x) = c_+(\alpha,\theta)_x\, I_+^\alpha f(x) + c_-(\alpha,\theta)_x\, I_-^\alpha f(x) \tag{4}$$

where the real parameters α and θ are always restricted as follows

$$0 < \alpha \le 2,\ \alpha \ne 1,$$
$$|\theta| \le \min\{\alpha, 2-\alpha\},$$

and also the coefficients

$$\begin{aligned} c_+(\alpha,\theta) &= \frac{\sin\left(\frac{(\alpha-\theta)\pi}{2}\right)}{\sin(\alpha\pi)}, \\ c_-(\alpha,\theta) &= \frac{\sin\left(\frac{(\alpha+\theta)\pi}{2}\right)}{\sin(\alpha\pi)}. \end{aligned} \tag{5}$$

Using the Feller potential, Gorenflo and Mainardi (1998) defined the Riesz-Feller derivative

$$\begin{aligned} \frac{\partial^\alpha f(x)}{\partial |x|_\theta^\alpha} &= -{}_xI_\theta^{-\alpha} f(x) \\ &= -\left[c_+(\alpha,\theta)_x\, D_+^\alpha f(x) + c_-(\alpha,\theta)_x\, D_-^\alpha f(x)\right] \end{aligned}$$

where ${}_xD_\pm^\alpha f(x)$ are Weyl fractional derivatives defined as

$${}_xD_\pm^\alpha f(x) = \begin{cases} \pm\dfrac{d}{dx}\left[{}_xI_\pm^{1-\alpha} f(x)\right], & 0 < \alpha < 1, \\ \dfrac{d^2}{dx^2}\left[{}_xI_\pm^{2-\alpha} f(x)\right], & 1 < \alpha \le 2. \end{cases} \tag{6}$$

The Caputo fractional derivative is defined as

$$\frac{\partial^\beta u(t)}{\partial t^\beta} = \frac{1}{\Gamma(n-\beta)} \int\limits_0^t (t-\tau)^{n-\beta-1} \left(\frac{d}{d\tau}\right)^n u(\tau)\, d\tau, \tag{7}$$

where $0 < \beta \le n, n \in \mathbb{Z}$. Now, we can formulate our problem after these preliminaries.

3. FORMULATION OF THE MAIN PROBLEM

Let we consider the following space-time fractional anomalous diffusion problem:

$$\frac{\partial^\beta u(x,y,t)}{\partial t^\beta} = \frac{\partial^\alpha u(x,y,t)}{\partial |x|_{\theta_1}^\alpha} + \frac{\partial^\mu u(x,y,t)}{\partial |y|_{\theta_2}^\mu}, \tag{8}$$

$$u(x,y,0)=u_0\left(x,y\right), \tag{9}$$

$$\lim_{x,y\to\pm\infty}u(x,y,t)=0, \tag{10}$$

where $x,y\in\mathbb{R}$; β,α,μ are real parameters restricted as $0<\beta\leq 1$, $0<\alpha<1$, $1<\mu\leq 2$; the skewness parameters θ_1 $(\theta_1\leq\min\{\alpha,1-\alpha\})$ and θ_2 $(\theta_2\leq\min\{\mu,2-\mu\})$ are measures of the asymmetry of the probability distribution of a real-valued random variable among the x and y coordinate axes. Note that, many simplistic mathematical models are defined under the Gaussian (normal) distribution; i.e., the skewness parameter is zero. However in reality, random variables may not distribute symmetrically. Therefore, the behaviour of such anomalous diffusion problem differs with the changing of θ_1 and θ_2 parameters. We first assume that the solution and the initial condition functions can be expanded in the following complex Fourier series expansions respectively,

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}\left(t\right)e^{inx}e^{imy}, \tag{11}$$

$$u_0\left(x,y\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{0nm}e^{inx}e^{imy}, \tag{12}$$

where $i^2=-1$. Under these assumptions, we calculate the fractional derivative terms in the right hand side of Eq.(8), respectively, as follows: We start with the calculation of $\frac{\partial^\alpha u(x,y,t)}{\partial|x|^\alpha_{\theta_1}}$ term which dependent on x variable and $0<\alpha<1$. Let us remind the definition:

$$\frac{\partial^\alpha u(x,y,t)}{\partial\left|x\right|^\alpha_{\theta_1}}=-\left[c_+\left(\alpha,\theta_1\right)_{-\infty}D^\alpha_x u(x,y,t)\right.$$
$$\left.+c_-\left(\alpha,\theta_1\right)_x D^\alpha_{+\infty}u(x,y,t)\right] \tag{13}$$

where

$${}_{-\infty}D^\alpha_x u(x,y,t)=\frac{\partial}{\partial x}\left(\frac{1}{\Gamma\left(1-\alpha\right)}\int_{-\infty}^{x}\frac{u(\xi,y,t)}{\left(x-\xi\right)^\alpha}d\xi\right) \tag{14}$$

and

$${}_xD^\alpha_{+\infty}u(x,y,t)=-\frac{\partial}{\partial x}\left(\frac{1}{\Gamma\left(1-\alpha\right)}\int_{x}^{\infty}\frac{u\left(\xi,y,t\right)}{\left(\xi-x\right)^\alpha}d\xi\right) \tag{15}$$

are left and right side Weyl fractional derivatives. Now, substituting Eq.(11) into Eq.(14) we have

$${}_{-\infty}D^\alpha_x u(x,y,t)=$$
$$=\frac{\partial}{\partial x}\left(\frac{1}{\Gamma\left(1-\alpha\right)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}\left(t\right)e^{imy}\int_{-\infty}^{x}\frac{e^{in\xi}}{\left(x-\xi\right)^\alpha}d\xi\right)$$
$$=\frac{1}{\Gamma\left(1-\alpha\right)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}\left(t\right)e^{imy}\frac{d}{dx}\left(e^{inx}\int_{0}^{\infty}\frac{e^{-inr}}{r^\alpha}dr\right)$$
$$=\frac{1}{\Gamma\left(1-\alpha\right)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}\left(t\right)e^{imy}\frac{d}{dx}\left(e^{inx}\left(in\right)^{\alpha-1}\Gamma\left(1-\alpha\right)\right)$$
$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(in\right)^\alpha u_{nm}\left(t\right)e^{imy}e^{inx} \tag{16}$$

and with the similar manipulations,

$${}_xD^\alpha_{+\infty}u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(-in\right)^\alpha u_{nm}\left(t\right)e^{imy}e^{inx}.$$

Hence, for $0<\alpha<1$,

$$\frac{\partial^\alpha u(x,y,t)}{\partial\left|x\right|^\alpha_{\theta_1}}=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}n^\alpha\left\{c_+\left(\alpha,\theta_1\right)\left(i\right)^\alpha\right.$$
$$\left.+c_-\left(\alpha,\theta_1\right)\left(-i\right)^\alpha\right\}u_{nm}\left(t\right)e^{imy}e^{inx}. \tag{17}$$

Now, we obtain a similar computation of $\frac{\partial^\mu u(x,y,t)}{\partial|y|^\mu_{\theta_2}}$ for the case of $1<\mu\leq 2$. Therefore, we get

$${}_{-\infty}D^\mu_y u(x,y,t)=\frac{\partial^2}{\partial y^2}\left(\frac{1}{\Gamma\left(2-\mu\right)}\int_{-\infty}^{y}\frac{u(x,\eta,t)}{\left(x-\eta\right)^{\mu-1}}d\eta\right)$$
$$=\frac{\partial^2}{\partial y^2}\left(\frac{1}{\Gamma\left(2-\mu\right)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}\left(t\right)e^{inx}\int_{-\infty}^{y}\frac{e^{im\eta}}{\left(x-\eta\right)^{\mu-1}}d\eta\right)$$
$$=\frac{1}{\Gamma\left(2-\mu\right)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}\left(t\right)e^{inx}\frac{d^2}{dy^2}\left(e^{imy}\int_{0}^{\infty}\frac{e^{-imk}}{k^{\mu-1}}dk\right)$$
$$=\frac{1}{\Gamma\left(2-\mu\right)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{nm}\left(t\right)e^{inx}\frac{d^2}{dy^2}\left(e^{imy}\left(im\right)^{\mu-2}\Gamma\left(2-\mu\right)\right)$$
$$=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}m^\mu\left(i\right)^{\mu-2}u_{nm}\left(t\right)e^{inx}e^{imy} \tag{18}$$

and

$${}_yD^\mu_{+\infty}u(x,y,t)=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}m^\mu\left(-i\right)^{\mu-2}u_{nm}\left(t\right)e^{inx}e^{imy}.$$

Hence, we obtain

$$\frac{\partial^\mu u(x,y,t)}{\partial\left|y\right|^\mu_{\theta_2}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}m^\mu\left\{c_+\left(\mu,\theta_2\right)\left(i\right)^{\mu-2}\right.$$
$$\left.+c_-\left(\mu,\theta_2\right)\left(-i\right)^{\mu-2}\right\}u_{nm}\left(t\right)e^{imy}e^{inx} \tag{19}$$

Consequently, substituting Eqs.(17) and (19) into Eq.(8) we take the following time fractional differential equation

$$\frac{\partial^\beta u_{nm}\left(t\right)}{\partial t^\beta}=\left\{-n^\alpha\left[c_+\left(\alpha,\theta_1\right)\left(i\right)^\alpha+c_-\left(\alpha,\theta_1\right)\left(-i\right)^\alpha\right]\right.$$
$$\left.+m^\mu\left[c_+\left(\mu,\theta_2\right)\left(i\right)^{\mu-2}+c_-\left(\mu,\theta_2\right)\left(-i\right)^{\mu-2}\right]\right\}u_{nm}\left(t\right). \tag{20}$$

Therefore, we reduce the Eq.(8) to a fractional differential equation with one fractional term. To find the $u_{nm}\left(t\right)$, we apply Laplace transform to Eq.(20) and obtain

$$s^\beta u_{nm}\left(s\right)-s^{\beta-1}u_{nm}\left(0\right)+Au_{nm}\left(s\right)=0 \tag{21}$$

where

$$A=\left\{n^\alpha\left[c_+\left(\alpha,\theta_1\right)\left(i\right)^\alpha+c_-\left(\alpha,\theta_1\right)\left(-i\right)^\alpha\right]\right. \tag{22}$$
$$\left.-m^\mu\left[c_+\left(\mu,\theta_2\right)\left(i\right)^{\mu-2}+c_-\left(\mu,\theta_2\right)\left(-i\right)^{\mu-2}\right]\right\}.$$

Using inverse Laplace transform, Eq.(21) reduces to

$$u_{nm}\left(t\right)=u_{nm}\left(0\right)E_{\beta,1}\left(-At^\beta\right) \tag{23}$$

where $E_{\beta,1}\left(.\right)$ is well known Mittag-Leffler function. The Fourier coefficients of the Eq.(12) can be found by

$$u_{0nm}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}u_0\left(x,y\right)e^{-inx}e^{-imy}dxdy, \tag{24}$$

After some manipulations, we take $u_{nm}\left(0\right)=u_{0nm}$ and also $u_{nm}\left(t\right)=u_{0nm}E_{\beta,1}\left(-At^\beta\right)$. Now, we can rewrite the solution series after these computations:

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}u_{0nm}(0)\,E_{\beta,1}\left(-At^{\beta}\right)e^{inx}e^{imy}. \quad (25)$$

4. GRÜNWALD-LETNIKOV APPROXIMATION FOR NUMERICAL SOLUTION

In this section, we show the numerical solution of the problem by applying GL approximation for Caputo derivative. Let we first give the relation between the left RL and Caputo definitions:

$$_{a}D_{t}^{\beta}u(t)= {}_{a}^{C}D_{t}^{\beta}u(t)+\sum_{r=0}^{m-1}\frac{d^{r}}{dt^{r}}u(t)\Big|_{x=a}\frac{(t-a)^{r-\beta}}{\Gamma(r-\alpha+1)}$$

where $m\in\mathbb{N}$, $m-1<\beta\leq m$, $a\in\mathbb{R}$. Note that, under the assumption $\left|\lim_{a\to-\infty}\frac{d^{r}}{dt^{r}}u(t)\Big|_{x=a}\right|<\infty$ for $r=0,1,...,m-1$, we have

$$_{-\infty}D_{x}^{\beta}u(t)= {}_{-\infty}^{C}D_{x}^{\beta}u(t).$$

It is also valid for the upper limit case and similar assumption as follows

$$_{x}D_{+\infty}^{\beta}u(t)= {}_{x}^{C}D_{+\infty}^{\beta}u(t).$$

We remind that the order of Caputo derivative is $0<\beta\leq 1$, the lower limit of derivative $a=0$ and so we obtain

$${}_{0}^{C}D_{x}^{\beta}u(t)= {}_{0}D_{x}^{\beta}u(t)\;-u(0)\frac{t^{-\beta}}{\Gamma(1-\beta)}.$$

It is well known that if a function has suitable properties, i.e. it has first-order continuous derivatives and its second-order derivative is integrable, the β-order derivatives of function in both RL and GL senses are the same. By this property, we discretize the RL operator applying GL definition to Eq.(20) and therefore, we take the approximation of Caputo derivative as

$${}_{0}^{C}D_{t}^{\beta}u_{nm}(t)\approx\frac{1}{h^{\beta}}\sum_{r=0}^{M}w_{r}^{(\beta)}u_{nm}(hM-rh)-u_{nm}(0)\frac{(hM)^{-\beta}}{\Gamma(1-\beta)} \quad (26)$$

where $M=\frac{t}{h}$ represents the number of sub-time intervals, h is step size and $w_{r}^{(\beta)}$ are the coefficients of GL formula:

$$w_{0}^{\beta}=1,\; w_{r}^{\beta}=\left(1-\frac{\beta+1}{r}\right)w_{(r-1)}^{\beta}. \quad (27)$$

Substituting Eq.(26) into Eq.(20) and after some arranging, we get

$$u_{nm}(hM)=\frac{1}{\left(\frac{1}{h^{\beta}}w_{0}^{(\beta)}+A\right)}\left\{u_{nm}(0)\frac{(hM)^{-\beta}}{\Gamma(1-\beta)}-\frac{1}{h^{\beta}}\sum_{r=1}^{M}w_{r}^{(\beta)}u_{nm}(hM-rh)\right\} \quad (28)$$

5. NUMERICAL EXAMPLE

In this section, we consider the following initial condition:

$$u(x,y,0)=\sinh(x+y)$$

In Fig.1, we first validate the efficiency of numerical method by comparison of analytical and numerical solutions for $x=\frac{\pi}{5}$, $y=\frac{\pi}{4}$, $t=5$, $h=0.01$ and $n=m=10$. It is clear from the figure that the analytical solution is in a good agreement with the numerical solution. Fig.2 shows the behaviour of problem under the variations of μ values for $x=\frac{\pi}{5}$, $y=\frac{\pi}{4}$, $t=5$, $h=0.01$, $\beta=1$, $\alpha=0.3$ and $\theta_1=0.3$. Similarly, Fig.3 shows the response of the problem for variable order of α for $t=5$, $\beta=0.5$, $\mu=1.5$ and $\theta_2=0.5$. Fig.4 indicates changing behaviours of problem with respect to the variations of α, β and μ parameters for $x=\frac{\pi}{5}$, $y=\frac{\pi}{4}$, $t=5$. In Fig.5, we get the three dimensional surface of the problem (8) with respect to x and t for $y=\frac{\pi}{4}$, $\beta=0.7$, $\alpha=0.5,\theta_1=0.5$ and $\mu=1.8,\theta_2=0.1$. Finally, we obtain the surface of the problem (8) with respect to x and y for $\beta=0.7$, $\alpha=0.5,\theta_1=0.5$ and $\mu=1.8,\theta_2=0.1$ and $h=0.01$ in Fig.6.

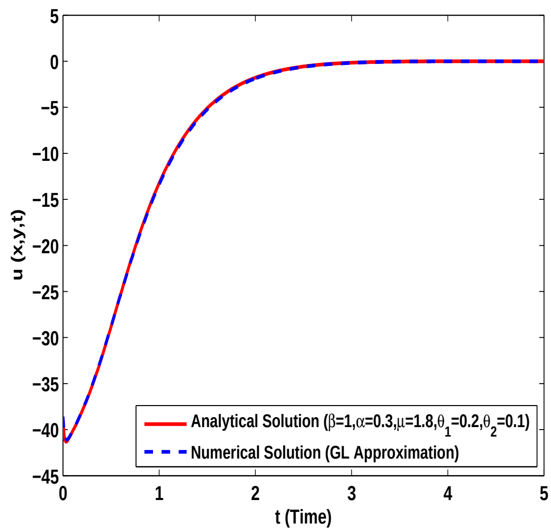

Fig. 1. Comparison of analytical and numerical solutions for $\beta=1$

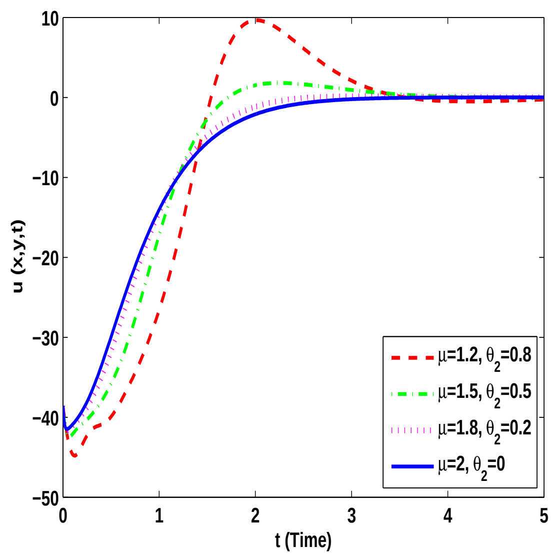

Fig. 2. Variations of μ parameter

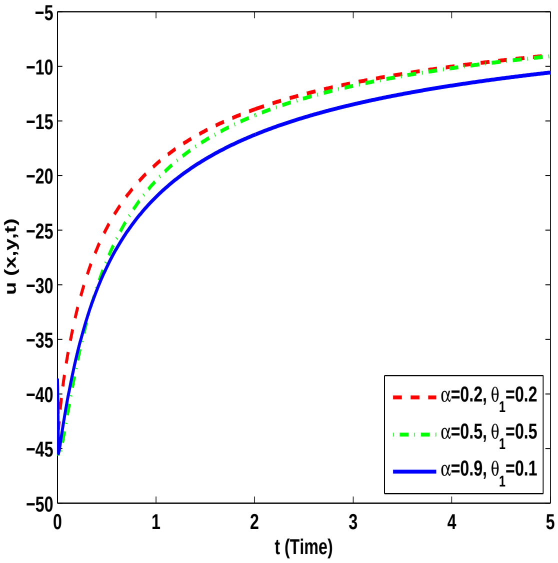

Fig. 3. Variations of α parameter

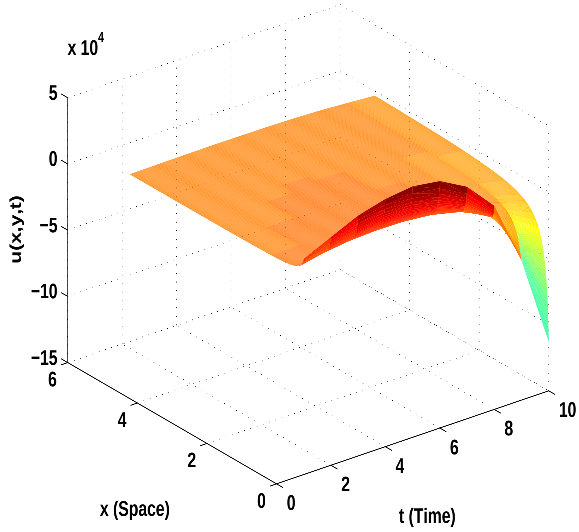

Fig. 5. Surface of the whole solution with respect to x and t

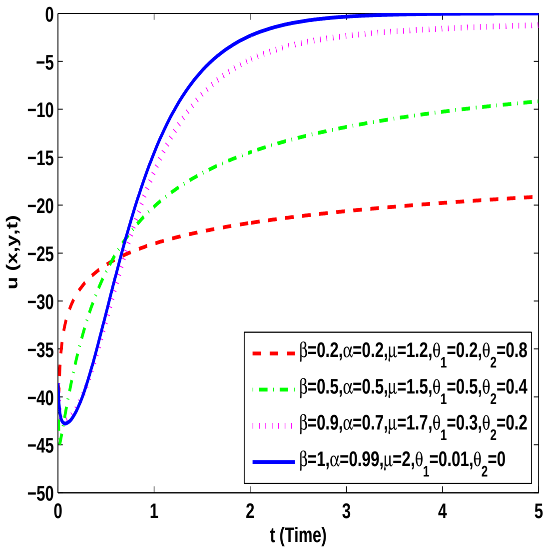

Fig. 4. Variations of β, α and μ parameters

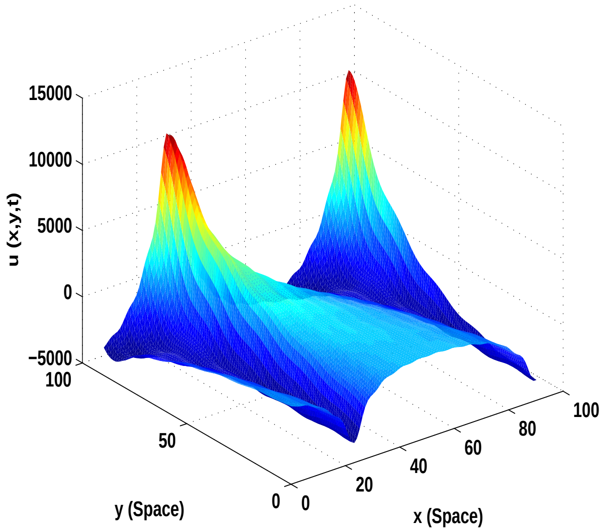

Fig. 6. Surface of the whole solution with respect to x and y

6. CONCLUSIONS

In this paper, we have defined a two-dimensional anomalous diffusion problem with time and space fractional derivative terms. These have been described in sense of Caputo and Riesz-Feller operators, respectively. We have purposed to find the exact and the numerical solutions of the problem under some assumptions. Therefore, we use Laplace and Fourier transforms for analytical solution and also prefer to apply GL definition. However, we first reduce the main problem to a fractional differential equation with time fractional term. By this way, we have obtained numerical results more easily. Finally, we apply the formulations to an example which are chosen arbitrarily. After that we present some figures under different considerations about variations of parameters. In addition, we deduce from the comparion of the analytical and the numerical solutions that the GL approximation can be applied successfully to such type of anomalous diffusion problems.

REFERENCES

A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* Elsevier B.V., The Netherlands, 2006.

K. S. Miller, B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* John Wiley&Sons, New York, NY, USA, 1993.

S. G. Samko, A. A. Kilbas, O. I. Marichev. *Fractional Integrals and Derivatives-Theory and Applications.* Gordon and Breach, Longhorne Pennsylvania, 1993.

I. Podlubny. *Fractional Differential Equations*. Academic Press, New York, 1999.

R. Metzler, J. Klafter. The random walk's guide to anomalous diffusion: A fractional dynamic approach. *Physics Reports*, volume 339, pages 1–77, 2000.

A. J. Turski, T. B. Atamaniuk, E. Turska. Application of fractional derivative operators to anomalous diffusion and propagation problems. *arXiv:math-ph/0701068v2*, 2007.

O. P. Agrawal. Response of a diffusion-wave system subjected to deterministic and stochastic fields. *Z. Angew. Math. Mech.*, volume 83 (4), pages 265-274, 2001.

O. P. Agrawal. Solution for a fractional diffusion-wave equation defined in a bounded domain. *Nonlinear Dynamics*, volume 29, pages 145-155, 2002.

M. A. E. Herzallah, A. M. A. El-Sayed, D. Baleanu. On the fractional-order diffusion-wave process. *Romanian Journal of Physics*, volume 55 (3-4), 2010.

F. Huang, F. Liu. The space-time fractional diffusion equation with Caputo derivatives. *Appl. Math. & Computing*, volume 19 (1-2), pages 179–190, 2005.

F. Huang, F. Liu. The fundamental solution of the space-time fractional advection-dispersion equation. *J. Appl. Math. & Computing*, volume 18, pages 339–350, 2005.

T. A. M. Langlands. Solution of A Modified Fractional Diffusion Equation. *Physica A*, volume 367, pages 136–144, 2006.

I. M. Sokolov, A. V. Chechkin, A. J. Klafter. Distributed-order fractional kinetics. *Acta Physica Polonica B*, volume 35, pages 1323–1341, 2004.

A. I. Saichev, G. M Zaslavsky. Fractional kinetic equations: Solutions and applications. *Chaos*, volume 7 (4), pages 753–764, 1997.

F. Mainardi, Y. Luchko, G. Pagnini. The fundamental solution of the space-time fractional diffusion equations. *Fractional Calculus and Applied Analysis*, volume 4 (3), pages 153–192, 2001.

S. Shen, F. Liu. Error analysis of an explicit finite difference approximation for the space fractional diffusion equation with insulated ends. *ANZIAM J.*, volume 46, pages 871–887, 2005.

F. Liu, P. Zhuang, V. Anh, I. Turner, K. Burrage. Stability and convergence of the difference methods for the space-time fractional advection-diffusion equation. *Applied Mathematics and Computation*, volume 191, pages 12–20, 2007.

R. Lin, F. Liu, V. Anh, I. Turner. Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation. *Applied Mathematics and Computation*, volume 212, pages 435–445, 2009.

R. Gorenflo, F. Mainardi. Random walk models for space-fractional diffusion processes. *Fractional Calculus and Applied Analysis*, volume 1, pages 167–191, 1998.

R. Gorenflo, F. Mainardi. Approximation of Levy-Feller diffusion by random walk." *Journal for Analysis and its Applications*, volume 18, pages 231–146, 1999.

N. Özdemir, O. P. Agrawal, D. Karadeniz, B. B. İskender. Analysis of an axis-symmetric fractional diffusion-wave equation." *J. Phys. A: Math. Theor.*, volume 42, 355208, 2009.

N. Özdemir, D. Karadeniz. Fractional diffusion-wave problem in cylindrical coordinates." *Phys. Lett. A*, volume 372, pages 5968–5972, 2008.

M. Ciesielski, J. Leszczynski. Numerical treatment of an initial-boundary value problem for fractional partial differential equations. *Signal Processing*. volume 86, pages 2619–2631, 2006.

M. Ciesielski, J. Leszczynski. Numerical solutions to boundary value problem for anomalous diffusion equation with Riesz-Feller fractional operator. *Journal of Theoretical and Applied Mechanics*, volume 44 (2), pages 393-403, 2006.

Q. Liu, F. Liu, I. Turner, V. Anh. Approximation of the Levy-Feller advection-dispersion process by random walk and finite difference method. *Journal of Computational Physics*, volume 222, pages 57–70, 2007.

H., Zhang, F., Liu, V.V., Anh. Numerical approximation of Levy-Feller diffusion equation and its probability interpretation. *Journal of Computational and Applied Mathematics*, volume 206 (2), pages 1098–1115, 2007.

J.A.T., Machado. A probabilistic interpretation of the fractional-order differentiation. *Fractional Calculus & Applied Analysis*, volume 6 (1), pages 73-80, 2003.

W., Feller. On a generalization of Marcel Riesz' potentials and the semi-groups generated by them. *Meddeladen Lund Universitets Matematiska Seminarium*, Tome suppl.dedie a M. Riesz, Lund, pages 73–81, 1952.