Application of MHPM to an optimization problem governed by fractional order differential equations

Fırat Evirgen[1] * **Necati Özdemir**[2] **

* *Department of Mathematics, Faculty of Science and Arts, Balıkesir University, Balıkesir, TURKEY (e-mail:fevirgen@balikesir.edu.tr)*
** *e-mail:nozdemir@balikesir.edu.tr*

Abstract: In this paper, the Homotopy Perturbation Method is applied to a solve nonlinear programming problem which are governed by fractional order differential equations. We consider the fractional derivative in the Caputo sense. Furthermore, the Multistage strategy is used to investigate the relation between an equilibrium point of the fractional order system and an optimal solution of the nonlinear programming problem. Because, it gives the behavior of a dynamic system in arbitrary longtime interval while the standard Homotopy Perturbation Method gives the optimal solution just only in the neighborhood of the initial time. The ability of the method for obtaining approximate analytical solutions was shown by comparisons among the Multistage Homotopy Perturbation Method, the standard Homotopy Perturbation Method and the fourth order Runge-Kutta Method.

Keywords: Nonlinear programming, Penalty function, Dynamical system, Fractional derivative, Homotopy perturbation method, Multistage strategy.

1. INTRODUCTION

Optimization theory is aimed to find out the optimal solution of problems which are defined mathematically from a model arise in wide range of scientific and engineering disciplines. Many methods and algorithms have been developed for this purpose since 1940. The penalty function methods are classical methods for solving nonlinear programming (NLP) problem, see Luenberger (1973) and Sun and Yuan (2006) for details. Also, differential equation methods are alternative approaches to find solutions to these problems. In this type of methods an optimization problem is formulated as a system of ordinary differential equations (ODEs) so that the equilibrium point of this system converges to the local minimum of the optimization problem. The methods based on ODEs for solving optimization problems have been proposed by Arrow et al. (1958), Rosen (1961), Fiacco and Mccormick (1968), and Yamashita Yamashita (1976). Recently, Wang et al. (2008), Jin et al. (2007) and Özdemir and Evirgen (2009, 2010) have prepared a novel differential equation approach for solving optimization problems.

In last decades, fractional calculus has drawn a wide attention from many physicists and mathematicians, because of its interdisciplinary application and physical meaning, e.g. Miller and Ross (1993); Oldham and Spanier (1974); Podlubny (1999). Fractional calculus deals with the generalization of differentiation and integration of noninteger order. Several analytical and numerical methods have been proposed for solving Fractional Differential Equations (FDEs). The variational iteration method (VIM) was first introduced by He (1997), and applied to FDE He (1998). The Adomian Decomposition Method (ADM) Adomian (1988, 1994) is another approach for solving linear and nonlinear problems. Moreover, the Homotopy Perturbation Method (HPM) is another successful analytical approximate technique, which provides a solution to linear and nonlinear problems, see He (1999, 2000). The HPM yields a very rapidly convergent series solution, and usually a few iterations lead to very accurate approximation of the exact solution He (1999, 2000, 2003, 2006a,b, 2007). The reason of this success is mainly based on combination of the traditional perturbation method and homotopy techniques. Recently, some techniques are adapted to the HPM for getting the essential behavior of the differential equation system for large time t, such as Multistage and Padé approximants. The homotopy perturbation method is used to solve wide range of differential equations in literature. For example, Baleanu and et al. have solved linear and nonlinear Schrödinger equations by HPM, Baleanu et al. (2009). Chowdhury and Hashim (2009) have employed HPM for solving Klein Gordon equation. Furthermore, adaptation of HPM with multistage strategy for numerical and analytical solutions of the system of ODEs was introduced by Hashim and Chowdhury (2008). Applications of multistage HPM for solving chaotic systems and biochemical reaction model were illustrated in Chowdhury et al. (2008); Hashim et al. (2008); Yu and Li (2009) references therein. Additionally, Abdulaziz et al. (2008) used HPM for solving system of FDEs. Odibat and Momani (2008) presented HPM for fractional order quadratic Riccati differential equation and fractional partial differential equation in Momani and Odibat (2007).

⋆ Corresponding author, nozdemir@balikesir.edu.tr

This paper construct a system of FDEs which is purpose to solve NLP problem with equality constraints. In order to see the coincidence between the steady state solution of the system of FDEs and the optimal solution of the NLP problem in a long time t period, we used the Multistage strategy.

The paper is organized as follows. In Section 2, the fundamentals of optimization problem, fractional calculus and HPM are briefly reviewed. In Section 3, the Multistage HPM is adapted to the nonlinear system of FDEs for solving NLP problem. In Section 4, the applicability and efficiency of Multistage HPM is illustrated by comparison among traditional HPM and fourth order Runge-Kutta (RK4) method on some numerical examples. And finally some concluding remarks are given in Section 5.

2. PRELIMINARIES

2.1 Optimization Problem

Consider the nonlinear programming problem with equality constraints defined by

$$\begin{array}{l}\text{minimize} \quad f(x),\\ \text{subject to } x \in M\end{array} \tag{1}$$

with

$$M = \{x \in \mathbb{R}^n \,|h(x) = 0\},$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $h = (h_1, h_2, ..., h_p)^T : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ ($p \leq n$). It is assumed that the functions in the problem are at least twice continuously differentiable, that a solution exists, and that $\nabla h(x)$ has full rank. To obtain a solution of (1), the penalty function method solves a sequence of unconstrained optimization problems. A well-known penalty function for this problem is given by

$$F(x, \mu) = f(x) + \mu \frac{1}{\gamma} \sum_{l=1}^{p} (h_l(x))^{\gamma}, \; l = 1, 2, ..., p, \tag{2}$$

where $\gamma > 0$ is constant and $\mu > 0$ is an auxiliary penalty variable. The corresponding unconstrained optimization problem of (1) is defined as follows;

$$\min \; F(x, \mu) \quad s.t. \; x \in \mathbb{R}^n. \tag{3}$$

Further information about NLP problem can be found in Luenberger (1973) and, Sun and Yuan (2006).

2.2 Fractional Calculus

Now we will give some definitions and properties of the fractional calculus Miller and Ross (1993); Oldham and Spanier (1974); Podlubny (1999). We begin with the Riemann-Liouville definition of the fractional integral of order $\alpha > 0$, which is given as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \qquad x > 0,$$

where $\Gamma(.)$ is the gamma function.

Most commonly encountered fractional derivatives are Riemann-Liouville and Caputo fractional derivative. The definitions of these two derivatives are given as:

Riemann-Liouville fractional derivative (RLFD)

$$\begin{aligned} D^{\alpha} f(x) &= D^m \left(I^{m-\alpha} f(x)\right) \\ &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_0^x (x-t)^{m-\alpha-1} f(t) dt, \end{aligned}$$

Caputo fractional derivative (CFD)

$$\begin{aligned} {}^C D^{\alpha} \; f(x) &= I^{m-\alpha} \left(D^m f(x)\right) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \left(\frac{d}{dt}\right)^m f(t) dt, \end{aligned}$$

where $m - 1 < \alpha \leqslant m$ and $m \in \mathbb{N}$. Note that D^m is the usual integer differential operator of order m. Furthermore,

$$I^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{s=0}^{m-1} f^{(s)}(0^+) \frac{x^s}{s!}, \quad m-1 < \alpha \leqslant m \tag{4}$$

is satisfied.

3. HOMOTOPY PERTURBATION METHOD

The brief outline of HPM is given in general by He (1999, 2003). For convenience, consider the following nonlinear differential equation

$$L(u) + N(u) = f(r), \quad r \in \Omega, \tag{5}$$

with boundary condition

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \quad r \in \Gamma,$$

where L is a linear operator, while N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function. The He's homotopy perturbation technique defines the homotopy $v(r, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$\begin{aligned} H(v, p) = &(1-p)\left[L(v) - L(u_0)\right] \\ &+ p\left[L(v) - N(v) - f(r)\right] = 0, \end{aligned} \tag{6}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. The changing process of p from zero to unity is just that of $v(r, p)$ from u_0 to $u(r)$. The basic assumption is that the solution of (6) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ...$$

The approximate solution of nonlinear equation (5), therefore can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{7}$$

The convergence of the series (7) has been proved in He (2000, 2007) and the asymptotic behavior of the series is given in He (2006a,b).

3.1 The Runge-Kutta Method

The Runge-Kutta method is one of the well known numerical methods for differential equations. The fourth order Runge-Kutta method computes the approximate solutions of the problem $x' = f(t, x)$ by the following iterative equations:

$$
\begin{aligned}
x_{n+1} &= x_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right),\\
k_1 &= f\left(t_n, x_n\right),\\
k_2 &= f\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_1\right),\\
k_3 &= f\left(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_2\right),\\
k_4 &= f\left(t_n + h, x_n + hk_3\right),
\end{aligned}
$$

where h is the fixed step size $t_i - t_{i-1}$ and x_n is the estimated value of the solution at the time t_n.

4. MULTISTAGE HPM FOR SYSTEM OF FDES

In this section, we solve nonlinear programming problems which are governed by a system of fractional differential equations. Consider the unconstrained optimization problem (3), an approach based on fractional dynamic system can be described by the following FDEs

$$ {}^{C}D^{\alpha}\ x(t) = -\nabla_x F(x, \mu), \tag{8} $$

subjected to the initial conditions

$$ x(t_0) = c_i, \quad i = 1, 2, ..., n $$

where ${}^{C}D^{\alpha}$ is the fractional derivative in Caputo sense of x of order α $(0 < \alpha \leqslant 1)$.

Note that, a point x_e is called an equilibrium point of (8) if it satisfies the right hand side of the Eq.(8). For convenience of reader, we reformulate the fractional dynamic system (8) as follow

$$ {}^{C}D^{\alpha}\ x_i(t) = g_i(t, \mu, x_1, x_2, ..., x_n), \ \ i = 1, 2, ..., n. \tag{9} $$

The steady state solution of the nonlinear system of FDEs (9) must be coincided with local optimal solution of the NLP problem (1).

In order to find the solution of the system (9), we use Multistage HPM. Because the multistage strategy is provided to reach steady state solution in whole time horizon rather than traditional HPM. According to the Eq.(6), we construct the following Homotopy

$$ {}^{C}D^{\alpha}\ x_i(t) = pg_i(t, \mu, x_1, x_2, ..., x_n), \tag{10} $$

where $i = 1, 2, ..., n$ and $p \in [0, 1]$. If $p = 0$, Eq.(10) becomes the linear equation

$$ {}^{C}D^{\alpha}\ x_i(t) = 0, $$

and when $p = 1$, the homotopy (10) turns out to be the original system given in (9).

We assume that the System (9) is defined on the time interval $t \in [0, T]$. We divide the time interval into N equal length subintervals $\Delta T = T_j - T_{j-1},\ j = 1, 2, ..., N$ with $T_0 = 0$ and $T_N = T$. Using the parameter p, we expand the solution x_i in the following form:

$$ x_i\left(t\right) = x_{i,0}\left(t\right) + px_{i,1}\left(t\right) + p^2 x_{i,2}\left(t\right) + ..., \quad i = 1, 2, 3, ..., n. \tag{11} $$

Also, we take the initial approximations as below

$$ x_{1,0}\left(t\right) = x_1\left(t^*\right),\ x_{2,0}\left(t\right) = x_2\left(t^*\right), ..., x_{n,0}\left(t\right) = x_n\left(t^*\right), \tag{12} $$

where t^* is the left end point of each subinterval and initial conditions as

$$
\begin{gathered}
x_{1,1}\left(t^*\right) = 0,\ x_{2,1}\left(t^*\right) = 0, ..., x_{n,1}\left(t^*\right) = 0\\
\vdots\\
x_{1,K}\left(t^*\right) = 0,\ x_{2,K}\left(t^*\right) = 0, ..., x_{n,K}\left(t^*\right) = 0\\
\vdots
\end{gathered}
$$

Substituting Eq.(11) into (10), and equating the coefficients of the terms with identical power of p, we get

$$
\begin{aligned}
&p^0:\ {}^{C}D^{\alpha}\ x_{i,0}(t) = 0\\
&p^1:\ {}^{C}D^{\alpha}\ x_{i,1}(t) = g_{i,1}(t, \mu, x_{1,0}, ..., x_{n,0})\\
&\qquad\vdots\\
&p^K:\ {}^{C}D^{\alpha}\ x_{i,K}(t) = g_{i,K}(t, \mu, x_{1,0}, ..., x_{n,1}; x_{1,1}, ..., x_{n,1}; ...;\\
&\qquad x_{1,K-1}, ..., x_{n,K-1})
\end{aligned}
\tag{13}
$$

where $i = 1, 2, 3, ..., n$ and the function $g_{i,1}, g_{i,2}, ...$ satisfy the following equation:

$$
\begin{aligned}
&g_i(t, \mu, x_{1,0} + px_{1,1} + \cdots, x_{2,0} + px_{2,1} + \cdots, x_{n,0} + px_{n,1} + \cdots)\\
&= g_{i,1}(t, \mu, x_{1,0}, ..., x_{n,0}) + pg_{i,2}(t, \mu, x_{1,0}, ..., x_{n,0}; x_{1,1}, ..., x_{n,1})\\
&+p^2 g_{i,3}(t, \mu, x_{1,0}, ..., x_{n,0}; x_{1,1}, ..., x_{n,1}; x_{1,2}, ..., x_{n,2}) + \cdots.
\end{aligned}
$$

For solving linear system (4), we apply the inverse operator I^{α} both side of equations. Therefore, the components $x_{i,k}$ $(i = 1, 2, ..., n\ ;\ k = 0, 1, 2, ...)$ of the multistage HPM can be determined. In order to carry out the iterations for every subinterval, we have to clarify initial approximations (12). For this purpose we set $t^* = t_0$. In Multistage HPM, the iterations provide appropriate value of solutions by means of the previous K-term approximations $\Phi_{i,K}$ of the preceding subinterval. Consequently, the approximate solution of (9) can be denoted as follow

$$ x_i(t) = \Phi_{i,K} = \sum_{k=0}^{K-1} x_{i,k}, \quad 1 \leqslant i \leqslant n. \tag{14} $$

Here the effectiveness and the applicability of the approach especially depend on choosing ΔT and the number of term in the approximate solution (14).

5. NUMERICAL IMPLEMENTATION

To illustrate the effectiveness of the Multistage HPM according to the HPM and fourth order Runge-Kutta method, some test problems are taken from Hock and Schittkowski (1981); Schittkowski (1987). Methods are coded in Maple and digits of the variables are set to 15 in all the calculations done in this paper.

Example 1 Consider the following nonlinear programming problem [Schittkowski (1987), Problem No: 216]

$$
\begin{aligned}
&\text{minimize} \quad f(x) = 100\left(x_1^2 - x_2\right)^2 + \left(x_1 - 1\right)^2\\
&\text{subject to}\ h(x) = x_1\left(x_1 - 4\right) - 2x_2 + 12 = 0.
\end{aligned}
\tag{15}
$$

The optimal solution is $x^* = (2, 4)^T$. For solving the above problem, we convert it to an unconstrained optimization problem with quadratic penalty function (2) for $\gamma = 2$, then we have

$$F(x,\mu) = 100\left(x_1^2 - x_2\right)^2 + (x_1-1)^2 + \frac{1}{2}\mu\left(x_1(x_1-4) - 2x_2 + 12\right)^2,$$

where $\mu \in \mathbb{R}^+$ is an auxiliary penalty variable. The corresponding nonlinear system of FDEs from (8) is defined as

$$\left.\begin{aligned} {}^C D^\alpha x_1(t) &= -400(x_1^2 - x_2)x_1 - 2(x_1-1) \\ &\quad - \mu(2x_1-4)(x_1^2 - 4x_1 - 2x_2 + 12) \\ {}^C D^\alpha x_2(t) &= 200(x_1^2 - x_2) + 2\mu(x_1^2 - 4x_1 - 2x_2 + 12), \end{aligned}\right\} \quad (16)$$

where $0 < \alpha \leqslant 1$. The initial conditions are $x_1(0) = 0$ and $x_2(0) = 0$. Utilizing the Homotopy (10) with auxiliary penalty variable $\mu = 800$ and step size $\Delta T = 0.00001$, the terms of the Multistage HPM solutions (14) are acquired. In Figure 1 and Figure 2, we show the approximate-exact solution x_1 and x_2 of the problem (15) for the derivative order $\alpha = 1$ and $\alpha = 0.9$. We see that for $\alpha = 1$ and $\alpha = 0.9$ our solutions obtained using the Multistage HPM are in good agreement with the RK4 method solution on $x^* = (2,4)^T$. Furthermore, numerical results in Table 1 and Table 2 show that the Multistage HPM for $\alpha = 0.9$ has better performance than for $\alpha = 1$.

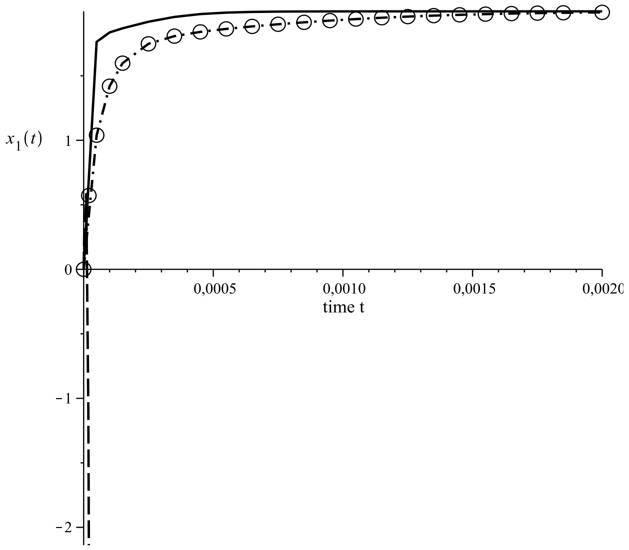

Fig. 1. Comparison of x_1; *Dash*: HPM for $\alpha = 0.9$, *Dashdot*: MHPM($\Delta T = 0.00001$) for $\alpha = 1$, *Solidline*: MHPM($\Delta T = 0.00001$) for $\alpha = 0.9$, ◯: RK4($\Delta T = 0.00001$) for $\alpha = 1$

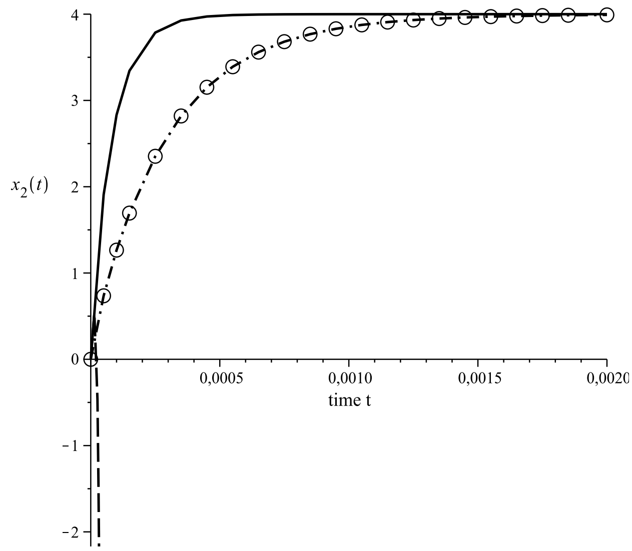

Fig. 2. Comparison of x_2; *Dash*: HPM for $\alpha = 0.9$, *Dashdot*: MHPM($\Delta T = 0.00001$) for $\alpha = 1$, *Solidline*: MHPM($\Delta T = 0.00001$) for $\alpha = 0.9$, ◯: RK4($\Delta T = 0.00001$) for $\alpha = 1$

Table 1. *A Comparison of $x(t)$ between HPM and MHPM for* $\alpha = 0.9$

	HPM ($\alpha = 0.9$)		MHPM ($\alpha = 0.9$)	
t	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0.000	0.0000	0.0000	0.0000	0.0000
0.001	$-0.69E+07$	$-0.11E+07$	1.9991	3.9996
0.002	$-0.84E+08$	$-0.14E+08$	1.9993	3.9998
0.003	$-0.36E+09$	$-0.62E+08$	1.9993	3.9998
0.004	$-0.10E+10$	$-0.17E+09$	1.9993	3.9998
0.005	$-0.23E+10$	$-0.39E+09$	1.9993	3.9998

Table 2. *A Comparison of $x(t)$ between MHPM and RK4 for* $\alpha = 1$

MHPM ($\alpha = 1$)		RK4 ($\alpha = 1$)		
t	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0.000	0.0000	0.0000	0.0000	0.0000
0.001	1.9338	3.8549	1.9338	3.8549
0.002	1.9916	3.9915	1.9916	3.9915
0.003	1.9986	3.9992	1.9986	3.9992
0.004	1.9993	3.9997	1.9992	3.9997
0.005	1.9994	3.9998	1.9993	3.9998

Example 2 Consider the equality constrained optimization problem [Schittkowski (1987), Problem No: 79]

$$\begin{aligned}\text{minimize} \quad & f(x) = (x_1 - 1)^2 + (x_1 - x_2)^2 \\ & \quad + (x_2 - x_3)^2 + (x_3 - x_4)^4 + (x_4 - x_5)^4 \\ \text{subject to} \quad & h_1(x) = x_1 + x_2^2 + x_3^3 - 2 - 3\sqrt{2} = 0 \\ & h_2(x) = x_2 - x_3^2 + x_4 + 2 - 2\sqrt{2} = 0 \\ & h_3(x) = x_1 x_5 - 2 = 0.\end{aligned} \tag{17}$$

The solution of (17) is

$$x^* \approx (1.191127, 1.362603, 1.472818, 1.635017, 1.679081)^T$$

and this is not an exact solution. The equality constrained optimization problem (17) is transformed to an unconstrained optimization problem by using quadratic penalty function (2) for $\gamma = 2$ as follows

$$F(x, \mu) = f(x) + \frac{1}{2}\mu \sum_{l=1}^{3} (h_l(x))^2,$$

where $\mu \in \mathbb{R}^+$ is an auxiliary penalty variable.
The corresponding nonlinear system of FDEs from (8) is defined as

$$^{C}D^{\alpha} x(t) = -\nabla f(x) - \mu \nabla h(x) h(x), \tag{18}$$

where $0 < \alpha \leqslant 1$. The initial condition is $x(0) = (2, 2, 2, 2, 2)^T$ that is not feasible. Using the homotopy (10) with auxiliary penalty variable $\mu = 75$ and step size $\Delta T = 0.0001$, the multistage HPM approximate-exact solutions (14) are obtained. In Table 3-7, the comparison of the solutions x_i, $i = 1, 2, 3, 4, 5$ between the HPM for $\alpha = 0.9$, the multistage HPM for $\alpha = 0.9$ and $\alpha = 1$ with the classical RK4 method are given, respectively. Here, the solutions continuously depend on the order of fractional derivative. Furthermore, our approximate solutions using the multistage HPM are in good agreement with the RK4 method solution and the optimal solution of the optimization problem (17).

Table 3. *A Comparison of $x(t)$ between HPM and MHPM for $\alpha = 0.9$*

	HPM ($\alpha = 0.9$)	MHPM ($\alpha = 0.9$)		
t	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0	2	2	2	2
2	$0.160E+8$	$0.620E+8$	1.198931	1.369223
10	$0.288E+9$	$0.112E+10$	1.191090	1.362530
15	$0.594E+9$	$0.230E+10$	1.191090	1.362530
20	$0.100E+10$	$0.388E+10$	1.191090	1.362530
30	$0.209E+10$	$0.811E+10$	1.191090	1.362530

Table 4. *A Comparison of $x(t)$ between MHPM and RK4 for $\alpha = 1$*

	MHPM ($\alpha = 1$)	RK4 ($\alpha = 1$)		
t	$x_1(t)$	$x_2(t)$	$x_1(t)$	$x_2(t)$
0	2	2	2	2
2	1.182161	1.352495	1.191010	1.359541
10	1.191050	1.362499	1.191082	1.362524
15	1.191084	1.362498	1.191090	1.362530
20	1.191082	1.362472	1.191090	1.362530
30	1.191113	1.362541	1.191090	1.362530

Table 5. *A Comparison of $x(t)$ between HPM and MHPM for $\alpha = 0.9$*

	HPM ($\alpha = 0.9$)		MHPM ($\alpha = 0.9$)	
t	$x_3(t)$	$x_4(t)$	$x_3(t)$	$x_4(t)$
0	2	2	2	2
2	$0.301E+9$	$-0.420E+7$	1.468744	1.616076
10	$0.546E+10$	$-0.756E+8$	1.472774	1.634738
15	$0.113E+11$	$-0.156E+9$	1.472774	1.634738
20	$0.191E+11$	$-0.263E+9$	1.472774	1.634738
30	$0.395E+11$	$-0.550E+9$	1.472774	1.634738

Table 6. *A Comparison of $x(t)$ MHPM with RK4 for $\alpha = 1$*

	MHPM ($\alpha = 1$)	RK4 ($\alpha = 1$)		
t	$x_3(t)$	$x_4(t)$	$x_3(t)$	$x_4(t)$
0	2	2	2	2
2	1.478320	1.661326	1.474039	1.641529
10	1.472792	1.634827	1.472778	1.634755
15	1.472786	1.634792	1.472774	1.634738
20	1.472798	1.634853	1.472774	1.634738
30	1.472765	1.634750	1.472774	1.634738

Table 7. *Comparison of $x(t)$ between HPM and MHPM with RK4 solutions for different value of α.*

	HPM ($\alpha = 0.9$)	MHPM ($\alpha = 0.9$)	MHPM ($\alpha = 1$)	RK4 ($\alpha = 1$)
t	$x_5(t)$	$x_5(t)$	$x_5(t)$	$x_5(t)$
0	2	2	2	2
2	$0.102E+7$	1.668076	1.691867	1.679209
10	$0.183E+8$	1.679130	1.679187	1.679140
15	$0.378E+8$	1.679130	1.679136	1.679130
20	$0.637E+8$	1.679130	1.679142	1.679130
30	$0.133E+9$	1.679130	1.679093	1.679130

6. CONCLUSIONS

In the present work, the Homotopy Perturbation Method (HPM) has been successfully used to obtain approximate analytical solutions of nonlinear programming (NLP) problems. Initially, the NLP problem is reformulated by a system of Fractional Differential Equations (FDEs). In order to see the essential behavior of the system of FDEs, the Multistage strategy adapted to the HPM. The numerical comparison among the fourth order Runge-Kutta (RK4), the Multistage HPM ($\alpha = 1$ and $\alpha = 0.9$) and HPM ($\alpha = 0.9$) shows that the Multistage HPM ($\alpha = 0.9$) performs rapid convergency to the optimal solutions of the optimization problems. Consequently, these results verify the efficiency of the Multistage HPM as a practical tool for solving NLP problem.

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