Robust Synchronization and Parameter Identification of Fractional-Order Unified Chaotic System

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Abstract: This paper addresses a robust synchronization of fractional order unified systems whilst some parameters need to be identified. Based on the Sliding mode theory, a control law is proposed to realize a robust synchronization and parameter identification of two fractional order unified systems especially in presence of discrepancy in the initial conditions. This is also done when the slave system is perturbed by the uncertainties in the dynamic and parameters of the master are made unknown. A novel switching surface is proposed to perform the task and raise the convergence rate of the error in the closed-loop sliding mode control. Also Unlike many well-known methods of the sliding mode control, no knowledge on the bound of uncertainty and disturbance is required. Simulation approach is given to assess validation and quality of the analysis and design.

Keywords: Fractional derivatives, unified system, Synchronization, Adaptive control, Sliding mode theory, Uncertainty.

1. INTRODUCTION

Chaotic systems have held researchers' interest in the past decades. Such nonlinear systems can model various natural and man-made systems, and are known to have great sensitivity to initial conditions. This means two system starting trajectories from their arbitrary and almost the same initial states could evolve in dramatically different fashions, and soon become uncorrelated and unpredictable. In recent years, a new direction of chaos research has emerged, in which fractional order calculus is applied to dynamic systems (Wang and Yu, 2008).

Fractional calculus is in essence as an extension of ordinary calculus, with almost 300-year-old history. In spite of the long history, the application of fractional calculus to physics and engineering are just a recent focus of interest (Podlubny, 1999). It has been found that the behaviour of many physical systems can be properly described by using the fractional order system theory. For example heat conduction (Jenson and Jeffreys, 1997), quantum evolution of complex systems (Kusnezov et al., 1999), and diffusion waves (EI-Sayed, 1996) are known systems governed by the fractional order equations. In fact, real world process generally or most likely is fractional order system (Torvik and Bagley, 1984). More recently, there is a new trend to investigate the control and dynamics of fractional order dynamical systems.

Ahmad (and Sprott, 2003) has shown that nonlinear chaotic systems can still show chaos when their models become fractional. Ahmad (and Harba, 2003) investigated chaos control for fractional chaotic systems, where controllers have been designed using ''backstepping'' method of nonlinear control design. Li (and Chen, 2004), found that chaos exists in the fractional order Chen system with order less than 3. Linear feedback control of chaos in this system is studied. In (Li et al., 2003) chaos synchronization of fractional order chaotic systems are studied.

A unified chaotic system is a chaotic system which depends on a parameter $\alpha \in [0,1]$. If $0 \le \alpha < 0.8$, the unified chaotic system reduces to the generalized Lorenz chaotic system; the unified chaotic system is reduced to the Lü chaotic system when $\alpha = 0.8$. $0.8 < \alpha \le 1$ makes the unified chaotic system the generalized Chen chaotic system. Several researchers have focused on control and synchronization of the unified chaotic system. Chen (and Lu, 2002) considered that the parameter of the two unified chaotic systems is unknown and an adaptive controller is used to achieve synchronization based on the Lyapunov stability theory. Chen (et al., 2004) investigated the stabilization and synchronization of the unified chaotic system via an impulsive control method. Lu (et. al., 2004) used linear feedback and adaptive control to synchronize an identical unified chaotic system with only one input controller. Ucar (et. al., 2006) used a nonlinear active controller to synchronize two coupled unified chaotic systems with three control inputs. Wang (and Liu, 2007) proved that the unified chaotic system is equivalent to a passive one which become asymptotically stabilized at equilibrium points. Wang (and Song, 2008) studied the synchronization problem of two identical unified chaotic systems using three different methods. They used a linear feedback controller, a nonlinear feedback method and an impulsive controller to synchronize the systems. In (Zribi et al., 2009) based on the sliding mode theory synchronization of two identical unified chaotic is discussed.

In this paper adaptive sliding mode control will be designed to synchronize two fractional order unified chaotic systems.

This will be done when parameters are unknown and need to be identified, especially when initial conditions of master and slave systems are different. This is also done when the slave system is perturbed by the uncertainties in the dynamic. Also Unlike many well-known methods of the sliding mode control, no knowledge on the bound of uncertainty and disturbance is required.

The paper is organized as follows: Section 2 describes the unified system. Fractional-order adaptive controller is proposed to synchronize and identify parameters of two the unified systems in section 3. Simulation study is given in section 4, to illustrate the effectiveness of the proposed controller. The paper will be concluded in section 5.

2. SYSTEM DESCRIPTION

Unified chaotic system is a system whose behaviour incorporates the behaviour of the chaotic Lorenz, Chen and the Lü systems. The unified chaotic system is governed by the following set of ordinary differential equations:

$$\begin{cases} \frac{dx}{dt} = (25\alpha+10)(y-x) \\ \frac{dy}{dt} = (28-35\alpha)x - xz + (29\alpha-1)y \\ \frac{dz}{dt} = xy - \frac{8+\alpha}{3}z \end{cases} \tag{1}$$

The states of the system (6) are *x*, *y* and *z* and the parameter of the system is α. Parameter α takes values in the range [0,1]. For $\alpha \in [0,1]$ the resulting system is chaotic. When $\alpha = 0.01$, the unified chaotic system represents the Lorenz chaotic attractor. It represents the Lü chaotic attractor when $\alpha = 0.8$. When $\alpha = 1$, it represents the Chen chaotic attractor (Bowong et al., 2006). Moreover, for $\alpha \in [0, 0.8)$, system (6) is called the general Lorenz system. System (6) is called the general Chen system when $\alpha \in (0.8, 1]$ (Femat et al., 2000).

Now, let us introduce the fractional version of equation (2). The standard derivatives in equation (1) are replaced by the fractional derivatives as follows:

$$\begin{cases} \frac{d^q x}{dt^q} = (25\alpha+10)(y-x) \\ \frac{d^q y}{dt^q} = (28-35\alpha)x - xz + (29\alpha-1)y \\ \frac{d^q z}{dt^q} = xy - \frac{8+\alpha}{3}z \end{cases} \tag{2}$$

where *q* is the fractional order and is subjected to $0 < q \le 1$.

Chaotic behaviour of fractional order unified systems (Chen, Lü and Lorenz-Like) for $q = 0.9, 0.95, 0.99$ are shown in (Matouk 2009). From fractional order unified chaotic system in (2) a generalized type can be given as follows:

$$\begin{cases} \frac{d^q x}{dt^q} = a(y-x) \\ \frac{d^q y}{dt^q} = bx - xz + cy \\ \frac{d^q z}{dt^q} = xy - dz \end{cases} \tag{3}$$

2.1 Fractional order Chen System

From Matouk (2009) the fractional order Chen system is given by:

$$\begin{cases} \frac{d^q x_1}{dt^q} = a_1(x_2 - x_1) \\ \frac{d^q x_2}{dt^q} = (c_1 - a_1)x_1 - x_1x_3 + c_1x_2 \\ \frac{d^q x_3}{dt^q} = x_1x_2 - b_1x_3 \end{cases} \tag{4}$$

The fractional order Chen system as master is represented from equation (4), where x_1, x_2, x_3 are the states and a_1, b_1, c_1 are the unknown constant parameters of the master dynamic.

A similar forced uncertain slave system can be written as:

$$\begin{cases} \frac{d^q y_1}{dt^q} = a_2(t)(y_2 - y_1) + \Delta f_1(y_1, y_2, y_3) + u_1 \\ \frac{d^q y_2}{dt^q} = (c_2(t) - a_2(t))y_1 - y_1y_3 + c_2(t)y_2 + \Delta f_2(y_1, y_2, y_3) + u_2 \\ \frac{d^q y_3}{dt^q} = y_1y_2 - b_2(t)y_3 + \Delta f_3(y_1, y_2, y_3) + u_3 \end{cases} \tag{5}$$

$a_2(t), b_2(t)$ and $c_2(t)$ are time dependent unknown parameters which must be identified through parameters of the master. $\Delta f_i(y_1, y_2, y_3)(i = 1,2,3)$ is an uncertain term, representing the unknown part of dynamic. The uncertainty is assumed upper bounded by a positive constant σ as $|\Delta f_i(y_1, y_2, y_3)| \le \sigma$.

Note that the slave system contains three input control signals. The control will be designed such that the master and the slave are synchronized after starting from different initial conditions. The error will be defined between the states of the master and the slave systems. The error dynamic will be written using equations (4) and (5) which is as follows:

$$\begin{cases} \frac{d^q e_1}{dt^q} = a_2(t)(e_2 - e_1) + \tilde{a}(x_2 - x_1) + \Delta f_1(y_1, y_2, y_3) + u_1 \\ \frac{d^q e_2}{dt^q} = c_2(t)(e_1 + e_2) + \tilde{c}(x_1 + x_2) - y_1e_3 - e_1x_3 + a_2(t)e_1 - x_1\tilde{a} + \Delta f_2(y_1, y_2, y_3) + u_2 \\ \frac{d^q e_3}{dt^q} = y_1e_2 + e_1x_2 - b_2(t)e_3 - x_3\tilde{b} + \Delta f_3(y_1, y_2, y_3) + u_3 \end{cases} \tag{6}$$

where $e_1 \triangleq y_1 - x_1$, $e_2 \triangleq y_2 - x_2$, $e_3 \triangleq y_3 - x_3$ are the states error and the tilda-terms show the deviation of parameters from their nominal values by:

$\tilde{a} = a_2(t) - a_1$, $\tilde{b} = b_2(t) - b_1$, $\tilde{c} = c_2(t) - c_1$.

2.2 Fractional order Lü System

From Matouk (2009) the fractional order Lü system is given by:

$$\begin{cases} \frac{d^q x_1}{dt^q} = a_1(x_2 - x_1) \\ \frac{d^q x_2}{dt^q} = -x_1 x_3 + c_1 x_2 \\ \frac{d^q x_3}{dt^q} = x_1 x_2 - b_1 x_3 \end{cases} \quad (7)$$

The master fractional order Lü system is represented from equation (7) whilst the forced uncertain slave system can be written as:

$$\begin{cases} \frac{d^q y_1}{dt^q} = a_2(t)(y_2 - y_1) + \Delta f_1(y_1, y_2, y_3) + u_1 \\ \frac{d^q y_2}{dt^q} = -y_1 y_3 + c_2(t) y_2 + \Delta f_2(y_1, y_2, y_3) + u_2 \\ \frac{d^q y_3}{dt^q} = y_1 y_2 - b_2(t) y_3 + \Delta f_3(y_1, y_2, y_3) + u_3 \end{cases} \quad (8)$$

Deduction of equations (7) from (8) yields the error dynamic by:

$$\begin{cases} \frac{d^q e_1}{dt^q} = a_2(t)(e_2 - e_1) + \tilde{a}(x_2 - x_1) + \Delta f_1(y_1, y_2, y_3) + u_1 \\ \frac{d^q e_2}{dt^q} = -y_1 e_3 - e_1 x_3 + c_2(t) e_2 + x_2 \tilde{c} + \Delta f_2(y_1, y_2, y_3) + u_2 \\ \frac{d^q e_3}{dt^q} = y_1 e_2 + e_1 x_2 - b_2(t) e_3 - x_3 \tilde{b} + \Delta f_3(y_1, y_2, y_3) + u_3 \end{cases} \quad (9)$$

In this paper, the goal is to design an adaptive sliding mode controller such that the resultant error of the robust synchronization and the parameter identification approaches zero. This means:

$$\lim_{t\to\infty} |e(t)| = \lim_{t\to\infty} |y(t) - x(t)| = 0$$

and

$$\lim_{t\to\infty} |\tilde{a}| = \lim_{t\to\infty} |a_2(t) - a_1| = 0$$

$$\lim_{t\to\infty} |\tilde{b}| = \lim_{t\to\infty} |b_2(t) - b_1| = 0$$

$$\lim_{t\to\infty} |\tilde{c}| = \lim_{t\to\infty} |c_2(t) - c_1| = 0$$

3. SLIDING MODE CONTROLLER

3.1 Design of the Controller for the Fractional order Chen System

A primary step in designing the sliding mode controller is to choose a sliding surface. An appropriate switching surface with integral operation is used such that the sliding motion on the manifold achieves desired properties. However, a sliding surface can be defined in the form of:

$$S(t) = \int_0^t (e_1(\tau) + e_2(\tau) + e_3(\tau)) d\tau + D^{q-1}(e_1 + e_2 + e_3) \quad (10)$$

Since the system dynamic is of the fractional, a similar fractional dynamic for surface is chosen. Due to complexity of the current synchronization task, an integral dynamic term is dedicated to be included in the surface. This will be shown providing a faster synchronization. The situation $S(t) = 0$ proves a stable dynamic for $e(t)$. Our aim is to design a controller to enable the system reaching to the sliding surface in a finite time. To ensure the occurrence of the sliding motion, the proposed control law and the adaptation mechanism are given by:

$$\begin{cases} u_1 = -e_1 - a_2(t)(e_2 - e_1) + k\,\mathrm{sgn}(S) \\ u_2 = -e_2 - c_2(t)(e_2 + e_1) + a_2(t) e_1 + y_1 e_3 + e_1 x_3 + k\,\mathrm{sgn}(S) \\ u_3 = -e_3 - y_1 e_2 - e_1 x_2 + b_2(t) e_3 + k\,\mathrm{sgn}(S) \end{cases} \quad (11)$$

$$\begin{cases} \dot{a}_2(t) = -S(x_2 - 2x_1) \\ \dot{b}_2(t) = S x_3 \\ \dot{c}_2(t) = -S(x_2 + x_1) \end{cases} \quad (12)$$

Where k is the reaching gain, achieved by following adaptive law:

$$\dot{k} = -3\gamma |S| \ , \ k(0) = \hat{k} \quad (13)$$

where γ is a positive constant number.

Lemma (Barbalat lemma, Khalil, 1992). If $\omega : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function for $t \geq 0$ and if the limit of the integral

$$\lim_{t\to\infty} \int_0^t \omega(\lambda) d\lambda$$

exists and is finite, then

$$\lim_{t\to\infty} \omega(t) = 0$$

Theorem 1. Consider the error dynamic (6) with unknown parameters and disturbance uncertainties. This system is controlled by an adaptive sliding mode controller (11) together with the adaptation mechanism (12). Then the state error trajectory converges to the sliding surface $S(t) = 0$.

Proof. Consider the following Lyapunov function as:

$$V = \frac{1}{2} S^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2) + \frac{1}{2\gamma}(\hat{k} + k)^2$$

Then, the first derivative is:

$$\dot V = S\dot S + \dot a_2(t)\tilde a + \dot b_2(t)\tilde b + \dot c_2(t)\tilde c + \frac{1}{\gamma}(\hat k + k)(\dot k) \qquad (14)$$

$$= S\begin{bmatrix} e_1+e_2+e_3+a_2(t)(e_2-e_1)+\tilde a(x_2-x_1) \\ +\Delta f_1(y_1+y_2+y_3)+u_1+c_2(e_2+e_1) \\ +\tilde c(x_2+x_1)-a_2e_1-x_1\tilde a-y_1e_3-e_1x_3 \\ +\Delta f_2(y_1+y_2+y_3)+u_2+y_1e_2+x_2e_1-b_2e_3-x_3\tilde b \\ +\Delta f_3(y_1+y_2+y_3)+u_3 \end{bmatrix}$$

$$+\dot a_2(t)\tilde a + \dot b_2(t)\tilde b + \dot c_2(t)\tilde c + \frac{1}{\gamma}(\hat k + k)(\dot k)$$

Substitution of equations (11) and (12) into equation (14) achieves the derivative of the Lyapunov function as:

$$\dot V = S\begin{bmatrix} 3k\,\mathrm{sgn}(S)+\Delta f_1(y_1,y_2,y_3)+\Delta f_2(y_1,y_2,y_3) \\ +\Delta f_3(y_1,y_2,y_3) \end{bmatrix} + \frac{1}{\gamma}(\hat k + k)(\dot k) \qquad (15)$$

$$\le 3k|S| + 3\sigma|S| + \frac{1}{\gamma}(\hat k + k)(\dot k)$$

$$\le 3k|S| + 3\sigma|S| + 3\hat k|S| - 3\hat k|S| + \frac{1}{\gamma}(\hat k + k)(\dot k)$$

$$\le 3(\sigma - \hat k)|S| + (k+\hat k)(\frac{1}{\gamma}\dot k + 3|S|)$$

From equation (13) and (15) we achieve:

$$\dot V = S\dot S \le 3(\sigma - \hat k)|S| \qquad (16)$$

It is clear that the scalar $\hat k$ can be chosen in such a way that the value of $(\sigma - \hat k)$ remains negative: (i.e., $(\sigma - \hat k) = -\eta$ where $\eta > 0$). And it is straight forward to verify that:

$$\dot V \le -3\eta|S| = -\omega(t) \le 0 \qquad (17)$$

where $\omega(t) = 3\eta|S|$. From equations (11) and (12) is concluded that the reaching condition $\dot V \le 0$ is always maintained. Since $\dot V$ is negative semi-definite, the origin of the error system is not asymptotically stable. In fact, as $\dot V \le 0$ then $S \in L_\infty$ and $\tilde a, \tilde b, \tilde c \in L_\infty$, accordingly $V(t) \in L_\infty$ (i.e. $S, \tilde a, \tilde b, \tilde c, V(t)$ are bounded). Then we have:

$$\int_0^t \omega(\lambda)d\lambda \le \int_0^t -\dot V d\lambda = V(0) - V(t) \le V(0)$$

As t goes infinite, the above integral is always less than or equal to $V(0)$. Since $V(0)$ is positive and finite,

$$\lim_{t\to\infty}\int_0^t \omega(\lambda)d\lambda$$

exists and is finite. Thus according to the Barbalat's lemma we obtain:

$$\lim_{t\to\infty}\omega(t) = \lim_{t\to\infty} 3\eta|S| = 0 \qquad (18)$$

Since η is greater than zero, (18) implies $S = 0$

Hence the proof is completely achieved. ■

3.2 Design of the Controller for the Fractional order Lü System

The same switching surface is used when the proposed control law and the adaptation mechanism are given by:

$$\begin{cases} u_1 = -e_1 - a_2(t)(e_2-e_1) + k\,\mathrm{sgn}(S) \\ u_2 = -e_2 - c_2(t)e_2 + a_2(t)e_1 + y_1e_3 + e_1x_3 + k\,\mathrm{sgn}(S) \\ u_3 = -e_3 - y_1e_2 - e_1x_2 + b_2(t)e_3 + k\,\mathrm{sgn}(S) \end{cases} \qquad (19)$$

$$\begin{cases} \dot a_2(t) = -S(x_2 - x_1) \\ \dot b_2(t) = Sx_3 \\ \dot c_2(t) = -Sx_2 \end{cases} \qquad (20)$$

Similar from previous section k is the reaching gain, achieved according of equation (13).

Theorem2. Consider the error dynamic in (9) with unknown parameters and disturbance uncertainties. The state error trajectory converges to the sliding surface $S(t) = 0$ if the sliding mode control law and the adaptation mechanism in equations (19) and (20) are applied.

Proof. Candidate the Lyapunov function as:

$$V = \frac{1}{2}S^2 + \frac{1}{2}(\tilde a^2 + \tilde b^2 + \tilde c^2) + \frac{1}{2\gamma}(\hat k + k)^2$$

The time derivative of V is obtained by:

$$\dot V = S\dot S + \dot a_2(t)\tilde a + \dot b_2(t)\tilde b + \dot c_2(t)\tilde c \qquad (21)$$

$$= S\begin{bmatrix} e_1+e_2+e_3+a_2(t)(e_2-e_1)+\tilde a(x_2-x_1) \\ +\Delta f_1(y_1+y_2+y_3)+u_1+c_2e_2 \\ +\tilde c x_2 - y_1e_3 - e_1x_3 + \Delta f_2(y_1+y_2+y_3)+u_2 \\ +y_1e_2+x_2e_1-b_2e_3-x_3\tilde b+\Delta f_3(y_1+y_2+y_3)+u_3 \end{bmatrix}$$

$$+\dot a_2(t)\tilde a + \dot b_2(t)\tilde b + \dot c_2(t)\tilde c + \frac{1}{\gamma}(k+\hat k)(\dot k)$$

Replacing equations (19), (20) and (13) into equation (21) achieves the derivative of the Lyapunov function by:

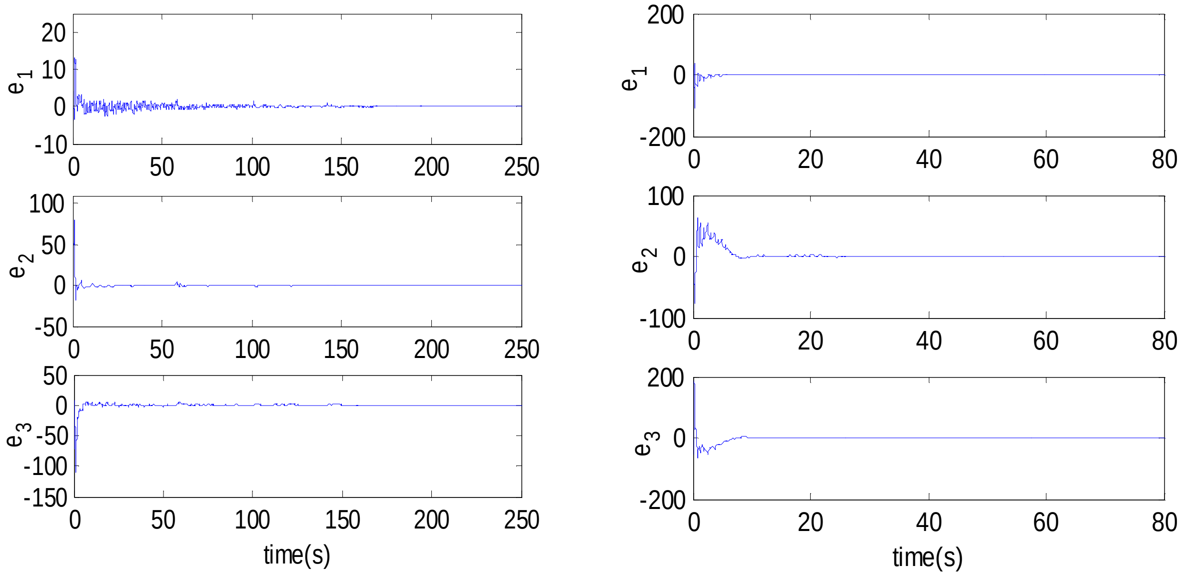
$$\dot V \le 3(\sigma - \hat k)|S| \qquad (22)$$

Again It is clear that the scalar $\hat k$ can be chosen in such a way that the value of $(\sigma - \hat k)$ remains negative.

From inequality (22), it is concluded that there exists a finite time t_1 such that for all $t \ge t_1$, the reach condition $\dot V \le 0$ is maintained. Similar from the previous section, from Barbalat's lemma $S = 0$ as $t \to \infty$ **.** Thus the proof is fully achieved. ■

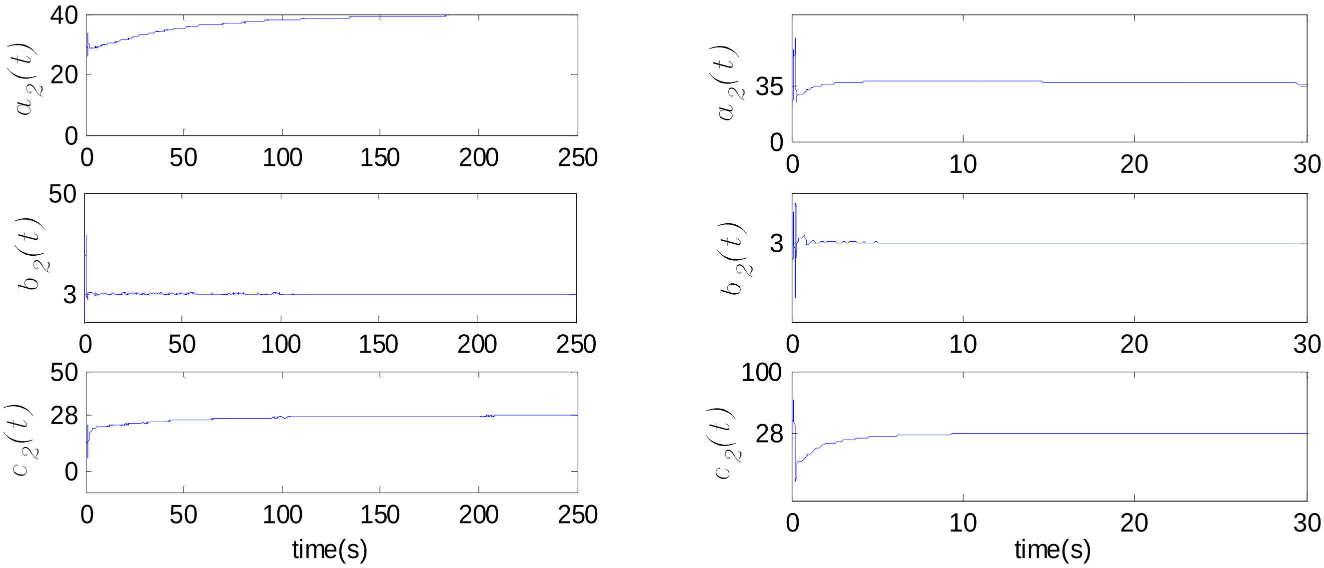
4. SIMULATION APPROACH

A simulation has been carried out using SIMULINK™, where the order is set to q=0.95. Adams method is used to solve the system of differential equations during the simulation. Initial conditions of states of master and slave are respectively selected as $(15,10,6)$ and $(12,8,7)$.



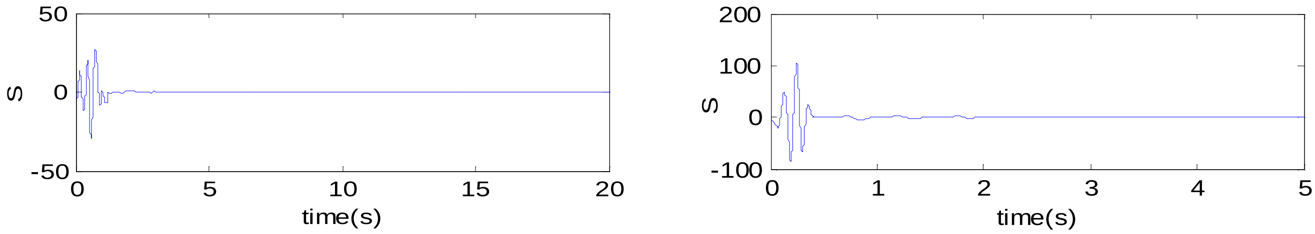
Chen system *Lü system*

Fig.1. Robust synchronization of fractional order unified chaotic systems



Chen system *Lü system*

Fig.2. Parameter identification of two unified systems when the master incorporates unknown parameters



Chen system *Lü system*

Fig. 3. Time response of the corresponding switching function $S(t)$

The master is also perturbed by an uncertainty term of:

$$\Delta f_i(y_1, y_2, y_3) = 0.5\sin(\sqrt{y_1^2 + y_2^2 + y_3^2}) \ \ (i = 1,2,3) \ .$$

In addition, the following adaptive law is used to update k .

$$\dot{k} = -6|S| \ \ , \ \ k(0) = 15$$

Also Fig. 3. show the corresponding switching function in equation (10).

4.1 Robust synchronization and Parameter identification of Chen Fractional order System

In this subsection robust synchronization and parameter identification of Chen fractional order system are concerned. Master system incorporates unknown constant parameters whilst the slave dynamic is perturbed by the uncertainties. To achieve robust synchronization and parameter identification, input controllers and the adaptation mechanism in equations (11) and (12) are respectively used. To obtain the Chen chaotic behavior, parameters in equation (4) is set to (Matouk, 2009):

$$a_1 = 40 \ , \ b_1 = 3, \ c_1 = 28$$

Result of synchronization and parameter identification are shown in figure (1) and (2) respectively.

4.2 Robust synchronization and Parameter identification of Lü Fractional order System

In this section a robust synchronization of Lü fractional order system is considered whilst parameters are also identified. It is also assumed the master uses unknown constant parameters and the slave is perturbed by the uncertainties. Robust synchronization and parameter identification will be achieved when input controllers and the adaptation mechanism in equations (19) and (20) are respectively used.

The Lü behaves chaotic when parameters in equation (7) are taken as (Matouk, 2009):

$$a_1 = 35 \ , \ b_1 = 3 \ , \ c_1 = 28$$

The result of synchronization and parameter identification are respectively shown in figure (1) and (2).

5. CONCLUSION

In this paper, adaptive sliding mode controller was used to synchronize a class of master–slave unified system through Lyapunov method. This is achieved by nonlinear inputs control when the system is also perturbed by the uncertainties. A novel switching surface is proposed to perform the task and raise the convergence rate of the error in the closed-loop sliding mode control and no knowledge on the bound of uncertainty and disturbance is required. The states error converges to zero as time tends to infinity. The simulation result verifies the capability of the proposed adaptive control scheme during the synchronization task through a parameter identification scheme. The synchronization is made possible for two identical systems with different initial conditions. The result also shows that the proposed control scheme is robust to bounded uncertainty.

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