Fractional Cauchy problems on bounded domains: survey of recent results

Erkan Nane *

* *Department of Mathematics and Statistics, 221 Parker Hall, Auburn University, Auburn, Alabama 36849, USA (e-mail: ezn0001@ auburn.edu)*

Abstract: In a fractional Cauchy problem, the usual first order time derivative is replaced by a fractional derivative. This problem was first considered by Nigmatullin (1986), and Zaslavsky (1994) in $\mathbb{R}^d$ for modeling some physical phenomena. The fractional derivative models time delays in a diffusion process. We will give a survey of the recent results on the fractional Cauchy problem and its generalizations on bounded domains $D \subset \mathbb{R}^d$ obtained in Meerschaert et al. (2009, 2010). We also study the solutions of fractional Cauchy problem where the first time derivative is replaced with an infinite sum of fractional derivatives. We point out a connection to eigenvalue problems for the fractional time operators considered. The solutions to the eigenvalue problems are expressed by Mittag-Leffler functions and its generalized versions. The stochastic solution of the eigenvalue problems for the fractional derivatives are given by inverse subordinators.

Keywords: Fractional diffusion, distributed-order Cauchy problems, Caputo fractional derivative, stochastic solution, uniformly elliptic operator, bounded domain, boundary value problem, Mittag-Leffler function, hitting time process.

1. INTRODUCTION

A celebrated paper of Einstein (1906) established a mathematical link between random walks, the diffusion equation, and Brownian motion. The scaling limits of a simple random walk with mean zero, finite variance jumps yields a Brownian motion. The probability densities of the Brownian motion variables solve a diffusion equation, and hence we refer to the Brownian motion as the stochastic solution to the diffusion (heat) equation. The diffusion equation is the most familiar Cauchy problem. The general abstract Cauchy problem is $\partial_t u = Lu$, where $u(t)$ takes values in a Banach space and L is the generator of a continuous semigroup on that space, see Arendt et al. (2001). If L generates a Markov process, then we call this Markov process a stochastic solution to the Cauchy problem $\partial_t u = Lu$, since its probability densities (or distributions) solve the Cauchy problem. This point of view has proven useful, for instance, in the modern theory of fractional calculus, since fractional derivatives are generators of certain (α-stable) stochastic processes, see Meerschaert and Sheffler (2004).

Fractional derivatives are almost as old as their more familiar integer-order counterparts, see Miller and Ross (1993); Samko et al. (1993). Fractional diffusion equations have recently been applied to problems in physics, finance, hydrology, and many other areas, see Gorenflo and Mainardi (2003); Kochubei (1990); Metzler and Klafter (2004); Scalas (2004). Fractional space derivatives are used to model anomalous diffusion or dispersion, where a particle plume spreads at a rate inconsistent with the classical Brownian motion model, and the plume may be asymmetric. When a fractional derivative replaces the second derivative in a diffusion or dispersion model, it leads to enhanced diffusion (also called superdiffusion). Fractional time derivatives are connected with anomalous subdiffusion, where a cloud of particles spreads more slowly than a classical diffusion. Fractional Cauchy problems replace the integer time derivative by its fractional counterpart: $\partial_t^\beta u = Lu$. Here, $\partial_t^\beta g(t)$ indicates the Caputo fractional derivative in time, the inverse Laplace transform of $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$, where $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$ is the usual Laplace transform, see Caputo (1967). Nigmatullin (1986) gave a physical derivation of the fractional Cauchy problem, when L is the generator of some continuous Markov process $\{Y(t)\}$ started at $x = 0$. The mathematical study of fractional Cauchy problems was initiated by Kochubei (1989, 1990); Schneider and Wyss (1989). The existence and uniqueness of solutions was proved in Kochubei (1989, 1990). Fractional Cauchy problems were also invented independently by Zaslavsky (1994) as a model for Hamiltonian chaos.

Stochastic solutions of fractional Cauchy problems are subordinated processes. If $X(t)$ is a stochastic solution to the Cauchy problem $\partial_t u = Au$, then under certain technical conditions, the subordinate process $X(E(t))$ is a stochastic solution to the fractional Cauchy problem $\partial_t^\beta u = Au$, see Baeumer and Meerschaert (2001). Here, $E(t)$ is the inverse or hitting time process to a stable subordinator $D(t)$ with index $\beta \in (0,1)$. That is, $E(t) = \inf\{x > 0 : D(x) > t\}$, and $D(t)$ is a Lévy process (continuous in probability with independent, stationary increments) whose smooth probability density $f_{D(1)}(t)$ has Laplace transform $e^{-s^\beta} = \tilde{f}_{D(1)}(s)$, see Sato (1999). Just as Brownian motion is a scaling limit of a simple random walk, the stochastic solution to certain fractional Cauchy problems

are scaling limits of continuous time random walks, in which the independent identically distributed (iid) jumps are separated by iid waiting times, see Meerschaert et al. (2002). Fractional time derivatives arise from power law waiting times, where the probability of waiting longer than time $t > 0$ falls off like $t^{-\beta}$ for t large, see Meerschaert and Sheffler (2004). This is related to the fact that fractional derivatives are non-local operators defined by convolution with a power law, see Baeumer and Meerschaert (2001).

In some applications, the waiting times between particle jumps evolve according to a more complicated process that cannot be adequately described by a single power law. Then, a waiting time model that is conditionally power law leads to a distributed-order fractional derivative in time, defined by integrating the fractional derivative of order β against the probability distribution of the power-law index, see Meerschaert and Sheffler (2006). The resulting distributed-order fractional Cauchy problem provides a more flexible model for anomalous sub-diffusion. The Lévy measure of a stable subordinator with index β is integrated against the power law index distribution to define a subordinator $W(t)$. Its inverse $E(t)$ produces a stochastic solution $X(E(t))$ of the distributed-order fractional Cauchy problem on $\mathbb{R}^d$, when $X(t)$ solves the original Cauchy problem $\partial u/\partial t = Lu$.

2. STOCHASTIC SOLUTION OF HEAT EQUATION ON BOUNDED DOMAINS

Let D be a bounded domain in $\mathbb{R}^d$. We denote by $C^k(D), C^{k,\alpha}(D), C^k(\bar{D})$ the space of k-times differentiable functions in D, the space of k-times differential functions with k-th derivative is Hölder continuous of index α, and the space of functions that have all the derivatives up to order k extendable continuously up to the boundary ∂D of D, respectively. We refer to Meerschaert et al. (2009) for a detailed discussion of these spaces and concepts in this section.

Suppose that every point of ∂D is regular for D^C. The corresponding Markov process is a killed Brownian motion. We denote the eigenvalues and the eigenfunctions of the Laplacian $\Delta = \sum_{i=1}^{d} \partial_{x_i}^2$ by $\{\mu_n, \phi_n\}_{n=1}^{\infty}$, where $\phi_n \in C^{\infty}(D)$.

Remark 2.1. In particular, the initial function f regarded as an element of $L^2(D)$ can be represented as

$$f(x) = \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x); \quad \bar{f}(n) = \int_D \phi_n(x)f(x)dx. \quad (1)$$

The corresponding heat kernel is given by

$$p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\mu_n t}\phi_n(x)\phi_n(y).$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$. In this case, the semigroup given by $T_D(t)f(x) = E_x[f(X(t))I(t < \tau_D(X))]$

$$= \sum_{n=1}^{\infty} e^{-\mu_n t}\phi_n(x)\bar{f}(n) = \int_D p_D(t, x, y)f(y)dy$$
$$(2)$$

solves the heat equation in D with Dirichlet boundary conditions:

$$\partial_t u(t, x) = \Delta u(t, x), \quad x \in D, \ t > 0,$$
$$u(t, x) = 0, \quad x \in \partial D,$$
$$u(0, x) = f(x), \quad x \in D.$$

3. FRACTIONAL CAUCHY PROBLEM

Fractional derivatives in time are useful for physical models that involve sticking or trapping, see Meerschaert et al. (2002). They are closely connected to random walk models with long waiting times between particle jumps, see Meerschaert and Sheffler (2004). The fractional derivatives are essentially convolutions with a power law. Various forms of the fractional derivative can be defined, depending on the domain of the power law kernel, and the way boundary points are handled, see Miller and Ross (1993); Samko et al. (1993). The Caputo fractional derivative invented by Caputo (1967) is defined for $0 < \beta < 1$ as

$$\partial_t^\beta u(t, x) = \frac{1}{\Gamma(1 - \beta)} \int_0^t \partial_r u(r, x)\frac{dr}{(t - r)^\beta}. \quad (3)$$

Its Laplace transform

$$\int_0^\infty e^{-st}\partial_t^\beta u(t, x)\, ds = s^\beta \tilde{u}(s, x) - s^{\beta-1}u(0, x) \quad (4)$$

incorporates the initial value in the same way as the first derivative. The Caputo derivative is useful for solving differential equations that involve a fractional time derivative, see Gorenflo and Mainardi (2003); Podlubny (1999), because it naturally incorporates initial values.

Let $D \in \mathbb{R}^d$ be a bounded domain. In this section we will consider the fractional Cauchy problem:

$$\partial_t^\beta u(t, x) = \Delta u(t, x), \quad x \in D, \ t > 0; \quad (5)$$
$$u(t, x) = 0, \quad x \in \partial D \ t > 0;$$
$$u(0, x) = f(x), \quad x \in D.$$

To obtain a solution, let $u(t, x) = G(t)F(x)$ be a solution of (5). Substituting in the PDE (5) leads to

$$F(x)\partial_t^\beta G(t) = G(t)\Delta F(x)$$

and now dividing both sides by $G(t)F(x)$, we obtain

$$\frac{\partial_t^\beta G(t)}{G(t)} = \frac{\Delta F(x)}{F(x)} = -\mu.$$

That is,

$$\partial_t^\beta G(t) = -\mu G(t), \ t > 0; \quad (6)$$
$$\Delta F(x) = -\mu F(x) \ x \in D; \ F(x) = 0, \ x \in \partial D. \quad (7)$$

Eigenvalue problem (7) is solved by an infinite sequence of pairs (μ_n, ϕ_n), $n \geq 1$, where ϕ_n is a sequence of functions that form a complete orthonormal set in $L^2(D)$, $\mu_1 < \mu_2 \leq \mu_2 \leq \cdots$, and $\mu_n \to \infty$.

Using the μ_n determined by (7), we need to find a solution of (6) with $\mu = \mu_n$, which is the eigenvalue problem for the Caputo fractional derivative.

We next consider the eigenvalue problem for the Caputo fractional derivative of order $0 < \beta < 1$.

Lemma 3.1. Let $\lambda > 0$. The unique solution of the eigenvalue problem

$$\partial_t^\beta G(t) = -\lambda G(t), \quad G(0) = 1. \quad (8)$$

is given by the Mittag-Leffler function

$$G(t) = M_\beta(-\lambda t^\beta) = \sum_{n=0}^\infty \frac{(-\lambda t^\beta)^n}{\Gamma(1 + \beta n)} \qquad (9)$$

For a detailed study of the Mittag-Leffler type functions we refer the reader to the tutorial paper by Gorenflo and Mainardi (1997).

Therefore the solution to (6) is given by

$$G(t) = G_0(n)M_\beta(-\mu t^\beta),$$

where $G_0(n) = \bar{f}(n)$ is selected to satisfy the initial condition f. Therefore using this lemma, we obtain a formal solution of the fractional Cauchy problem (5) as

$$u(t, x) = \sum_{n=1}^\infty \bar{f}(n)M_\beta(-\mu_n t^\beta)\phi_n(x) \qquad (10)$$

Remark 3.2. The separation of variables technique works for a large class operators including uniformly elliptic operators L in divergence form, see Remark 4.4. For details see Meerschaert et al. (2009). Define a cube

$D = \{x = (x_1, x_2, \cdots, x_d) : 0 < x_i < M \text{ for all } 1 \le i \le d\}$.

The functions

$$\phi_n(x) = (2/M)^{d/2} \prod_{i=1}^d \sin(\pi n_i x_i/M)$$

parametrised by the multi-index of positive integers $n = \{n_1, n_2, \cdots, n_d\}$, form a complete orthonormal set of eigenfunctions of the Laplacian, with Dirichlet boundary conditions with corresponding eigenvalues

$$\mu_n = \pi^2 M^{-2}(n_1^2 + \cdots + n_d^2).$$

See, for example, Lemma 6.2.1 in Davies (1995). In this case the boundary of the cube domain is not smooth.

3.1 Stochastic solution.

Fractional time derivatives emerge in anomalous diffusion models, when particles wait a long time between jumps. In the standard model, called a continuous time random walk (CTRW), a particle waits a random time $J_n > 0$ and then takes a step of random size Y_n. Suppose that the two sequences of i.i.d. random variables (J_n) and (Y_n) are independent. The particle arrives at location $X(n) = Y_1 + \cdots + Y_n$ at time $T(n) = J_1 + \cdots + J_n$. Since $N(t) = \max\{n \ge 0 : T(n) \le t\}$ is the number of jumps by time $t > 0$, the particle location at time t is $X(N(t))$. If $\mathbb{E}Y_n = 0$ and $\mathbb{E}[Y_n^2] < \infty$ then, as the time scale $c \to \infty$, the random walk of particle jumps has a scaling limit $c^{-1/2}X([ct]) \Rightarrow B(t)$, a standard Brownian motion. If $P(J_n > t) \sim ct^{-\beta}$ for some $0 < \beta < 1$ and $c > 0$, then the scaling limit $c^{-1/\beta}T([ct]) \Rightarrow D(t)$ is a strictly increasing stable Lévy process with index β, sometimes called a stable subordinator. The jump times $T(n)$ and the number of jumps $N(t)$ are inverses $\{N(t) \ge n\} = \{T(n) \le t\}$, and it follows that the scaling limits are also inverses, see Meerschaert and Sheffler (2004, Theorem 3.2): $c^{-\beta}N(ct) \Rightarrow E(t)$, where

$$E(t) = \inf\{\tau : D(\tau) > t\}, \qquad (11)$$

so that $\{E(t) \le \tau\} = \{D(\tau) \ge t\}$. A continuous mapping argument in Meerschaert and Sheffler (2004, Theorem 4.2) yields the CTRW scaling limit: Heuristically, since $N(ct) \approx$

$c^\beta E(t)$, we have $c^{-\beta/2}X(N([ct])) \approx (c^\beta)^{-1/2}X(c^\beta E(t)) \approx B(E(t))$, a time-changed Brownian motion. The density $u(t, x)$ of the process $B(E(t))$ solves a fractional Cauchy problem

$$\partial_t^\beta u(t, x) = \partial_x^2 u(t, x)$$

where the order of the fractional derivative equals the index of the stable subordinator. Roughly speaking, if the probability of waiting longer than time $t > 0$ between jumps falls off like $t^{-\beta}$, then the limiting particle density solves a diffusion equation that involves a fractional time derivative of the same order β.

The Laplace transform of $D(t)$ is given by

$$\mathbb{E}(e^{-sD(t)}) = \int_0^\infty e^{-sx} f_{D(t)}(x)dx = e^{-ts^\beta}.$$

The inverse to the stable subordinator $E(t)$ of $D(t)$ has density

$$\begin{aligned}
f_{E(t)}(l) &= \frac{\partial}{\partial l}P(E(t) \le l) = \frac{\partial}{\partial l}(1 - P(D(l) \le t)) \\
&= -\frac{\partial}{\partial l}\int_0^{\frac{t}{l^{1/\beta}}} f_{D(1)}(u)du \qquad (12) \\
&= (t/\beta)f_{D(1)}(tl^{-1/\beta})l^{-1-1/\beta}
\end{aligned}$$

using the scaling property of the density $f_{D(t)}(l) = t^{-1/\beta}f_{D(1)}(lt^{-1/\beta})$, see Bertoin (1996).

Using the representation (12) and taking Laplace transforms we can show that the unique solution of the eigenvalue problem (8) is also given by

$$G(t) = \int_0^\infty \exp(-l\lambda)f_{E(t)}(l)dl = \mathbb{E}(\exp(-\lambda E(t))) \quad (13)$$

see Meerschaert et al. (2002) for the details.

Now using (2) and (13) we can express the solution to (5) as

$$\begin{aligned}
u(t, x) &= \sum_{n=1}^\infty \bar{f}(n)M_\beta(-\mu_n t^\beta)\phi_n(x) \\
&= \sum_{n=1}^\infty \bar{f}(n)[\int_0^\infty \exp(-l\mu_n)f_{E(t)}(l)dl]\phi_n(x) \\
&= \int_0^\infty [\sum_{n=1}^\infty \bar{f}(n)\exp(-l\mu_n)\phi_n(x)]f_{E(t)}(l)dl \quad (14) \\
&= \int_0^\infty [T_D(l)f(x)]f_{E(t)}(l)dl \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_M > E(t))] \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_M(X(E)) > t)].
\end{aligned}$$

Remark 3.3. Meerschaert et al. (2009) established the conditions on the initial function f under which $u(t, x)$ is a classical solution (i.e., for each $t > 0$, $u(t, x) \in C^1(\bar{D}) \cap C^2(D)$ and for each $x \in D$, $u(t, x) \in C^1(0, \infty)$) of (5): that $\Delta f(x)$ has an eigenfunction expansion w.r.t. $\{\phi_n\}$ that is absolutely and uniformly convergent. The analytic expression in $(0, M) \subset \mathbb{R}$ above is due to Agrawal (2002).

4. DISTRIBUTED-ORDER FRACTIONAL CAUCHY PROBLEMS

Let μ be a finite measure with $supp\mu \subset (0, 1)$. We consider the distributed order-time fractional derivative

$$\mathbb{D}^{(\nu)}u(t,x) := \int_0^1 \partial_t^\beta u(t,x)\nu(d\beta), \ \nu(d\beta) = \Gamma(1-\beta)\mu(d\beta).$$
(15)

To ensure that $\mathbb{D}^{(\nu)}$ is well-defined, we impose the condition

$$\int_0^1 \frac{1}{1-\beta}\mu(d\beta) < \infty$$
(16)

as in Meerschaert and Sheffler (2006, Eq. (3.3)). Since $\Gamma(x) \sim 1/x$, as $x \to 0+$, this ensures that $\nu(d\beta)$ is a finite measure on $(0,1)$.

4.1 Eigenvalue problem: solution with waiting time process

Stochastic solution to the distributed-order fractional Cauchy problem is obtained by considering a a more flexible sequence of CTRW. At each scale $c > 0$, we are given i.i.d. waiting times (J_n^c) and i.i.d. jumps (Y_n^c). Assume the waiting times and jumps form triangular arrays whose row sums converge in distribution. Letting $X^c(n) = Y_1^c + \cdots + Y_n^c$ and $T^c(n) = J_1^c + \cdots + J_n^c$, we require that $X^c(cu) \Rightarrow A(t)$ and $T^c(cu) \Rightarrow W(t)$ as $c \to \infty$, where the limits $A(t)$ and $W(t)$ are independent Lévy processes. Letting $N_t^c = \max\{n \geq 0 : T^c(n) \leq t\}$, the CTRW scaling limit $X^c(N_t^c) \Rightarrow A(E_t^\nu)$ see, Meerschaert and Sheffler (2008, Theorem 2.1). A power-law mixture model for waiting times was proposed in Meerschaert and Sheffler (2006): Take an i.i.d. sequence of mixing variables (B_i) with $0 < B_i < 1$ and assume $P\{J_i^c > u|B_i = \beta\} = c^{-1}u^{-\beta}$ for $u \geq c^{-1/\beta}$, so that the waiting times are power laws conditional on the mixing variables. The waiting time process $T^c(cu) \Rightarrow W(t)$ a nondecreasing Lévy process, or subordinator, with $\mathbb{E}[e^{-sW(t)}] = e^{-t\psi_W(s)}$ and Laplace exponent

$$\psi_W(s) = \int_0^\infty (e^{-sx}-1)\phi_W(dx).$$
(17)

The Lévy measure

$$\phi_W(t,\infty) = \int_0^1 t^{-\beta}\mu(d\beta) = \int_0^1 \frac{t^{-\beta}}{\Gamma(1-\beta)}\nu(d\beta),$$
(18)

where μ is the distribution of the mixing variable, see Meerschaert and Sheffler (2006, Theorem 3.4 and Remark 5.1). A computation in Meerschaert and Sheffler (2006, Eq. (3.18)) using $\int_0^\infty (1-e^{-st})\beta t^{-\beta-1}dt = \Gamma(1-\beta)s^\beta$ shows that

$$\psi_W(s) = \int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta) = \int_0^1 s^\beta \nu(d\beta).$$
(19)

Then $c^{-1}N_t^c \Rightarrow E^\nu(t)$, the inverse subordinator, see Meerschaert and Sheffler (2006, Theorem 3.10). The general infinitely divisible Lévy process limit $A(t)$ forms a strongly continuous convolution semigroup with generator L (e.g., see Arendt et al. (2001)) and the corresponding CTRW scaling limit $A(E^\nu(t))$ is the stochastic solution to the distributed-order fractional Cauchy problem Meerschaert and Sheffler (2006, Eq. (5.12)) defined by

$$\mathbb{D}^{(\nu)}u(t,x) = Lu(t,x).$$
(20)

Since $\phi_W(0,\infty) = \infty$ in (18), Theorem 3.1 in Meerschaert and Sheffler (2008) implies that the inverse subordinator

$$E^\nu(t) = \inf\{x > 0 : W(x) > t\}$$
(21)

has density

$$g(t,x) = \int_0^t \phi_W(t-y,\infty)P_{W(x)}(dy).$$
(22)

This same condition ensures also that $E^\nu(t)$ is almost surely continuous, since $W(t)$ jumps in every interval, and hence is strictly increasing. Further, it follows from the definition (21) that $E^\nu(t)$ is monotone nondecreasing.

We say that a function is a **mild solution** to a pseudo-differential equation if its transform solves the corresponding equation in transform space. The next Lemma follows easily by taking Laplace transforms.

Lemma 4.1. (Meerschaert et al. (2010)). For any $\lambda > 0$, $h(t,\lambda) = \int_0^\infty e^{-\lambda x}g(t,x)\,dx = \mathbb{E}[e^{-\lambda E^\nu(t)}]$ is a mild solution of

$$\mathbb{D}^{(\nu)}h(t,\lambda) = -\lambda h(t,\lambda); \ \ h(0,\lambda) = 1.$$
(23)

Kochubei (2009) considered the following: Let $\rho(\alpha)$ be a right continuous non-decreasing step function on $(0,1)$. Assume that ρ has two sequences of jump points, β_n and ν_n, $n = 0,1,2,\cdots$, where $\beta_n \to 0$, $\nu_n \to 1$, $\beta_0 = \nu_0 \in (0,1)$. Suppose also that the sequence $\{\beta_n\}$ is strictly decreasing and $\{\nu_n\}$ is strictly increasing. Let $\gamma_n^1 = (\rho(\beta_n) - \rho(\beta_n - 0))$ and $\gamma_n^2 = (\rho(\nu_n) - \rho(\nu_n - 0))$, and define the distributed order differential operator

$$\mathbb{D}^{(\rho)}u(t,x) = \sum_{n=0}^\infty \gamma_n^1 \partial_t^{\beta_n}u(t,x) + \sum_{n=0}^\infty \gamma_n^2 \partial_t^{\nu_n}u(t,x).$$
(24)

Since ρ is a finite measure we have

$$\sum_{n=1}^\infty \gamma_n^1 < \infty, \ \ \sum_{n=1}^\infty \gamma_n^2 < \infty.$$

Here the corresponding subordinator is the sum of infinitely many independent stable subordinators;

$$W_t = \sum_{n=0}^\infty (\gamma_n^1)^{1/\beta_n}\Gamma(1-\beta_n)W_t^{\beta_n}$$
$$+ \sum_{n=1}^\infty (\gamma_n^2)^{1/\nu_n}\Gamma(1-\nu_n)W_t^{\nu_n}$$
(25)

for independent stable subordinators $W_t^{\beta_n}$, $W_t^{\nu_n}$ for $n = 0,1,\cdots$.

Lemma 4.2. Let $E^\rho(t) = \inf\{x > 0 : \ W(x) > t\}$. Then $h(t,\mu) = \mathbb{E}(e^{-\mu E^\rho(t)})$ is the classical solution to the eigenvalue problem

$$\mathbb{D}^{(\rho)}h(t,\mu) = -\mu h(t,\mu), \ h(0,\mu) = 1.$$

Using inverse Laplace transforms Kochubei (2009) established the following representation of $h(t,\mu)$:

$$h(t,\mu) = \frac{\mu}{\pi}\int_0^\infty r^{-1}e^{-tr}\frac{H_1(r)}{H_2(r)}dr$$
(26)

where

$$H_1(r) = \sum_{n=0}^\infty \left[\gamma_n^1 r^{\beta_n}\sin(\pi\beta_n) + (\gamma_n^2)r^{\nu_n}\sin(\pi\nu_n)\right]$$

$$H_2(r) = \left\{\mu + \sum_{n=0}^\infty \left[\gamma_n^1 r^{\beta_n}\cos(\pi\beta_n) + \gamma_n^2 r^{\nu_n}\cos(\pi\nu_n)\right]\right\}^2$$
$$+ \left\{\sum_{n=0}^\infty \left[\gamma_n^1 r^{\beta_n}\sin(\pi\beta_n) + \gamma_n^2 r^{\nu_n}\sin(\pi\nu_n)\right]\right\}^2.$$
(27)

Let $D \subset \mathbb{R}^d$ be a bounded domain with $\partial D \in C^{1,\alpha}$ for some $0 < \alpha < 1$, and $D_\infty = (0,\infty) \times D$. We will write

$u \in C^k(\bar{D})$ to mean that for each fixed $t > 0$, $u(t, \cdot) \in C^k(\bar{D})$, and $u \in C_b^k(\bar{D}_\infty)$ to mean that $u \in C^k(\bar{D}_\infty)$ and is bounded.

Define
$$\mathcal{H}_\Delta(D_\infty) = \{u : D_\infty \to \mathbb{R} : \Delta u(t, x) \in C(D_\infty)\};$$
$$\mathcal{H}_\Delta^b(D_\infty) = \mathcal{H}_\Delta(D_\infty) \cap \{u : |\partial_t u(t, x)| \le k(t)g(x),$$
$$g \in L^\infty(D), \ t > 0\},$$

for some functions k and b satisfying the condition

$$b(\lambda) \int_0^1 \int_0^t \frac{k(s)ds}{(t-s)^\beta} d\mu(\beta) < \infty, \qquad (28)$$

for $t, \lambda > 0$ and

$$k(t) \sum_{n=1}^\infty b(\lambda_n)\bar{f}(n)|\phi_n(x)| < \infty. \qquad (29)$$

Theorem 1. Let $f \in C^1(\bar{D}) \cap C^2(D)$ for which the eigenfunction expansion (of Δf) with respect to the complete orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ converges uniformly and absolutely. Then the classical solution to the distributed-order fractional Cauchy problem

$$\mathbb{D}^{(\rho)}u(t, x) = \Delta u(t, x), \quad x \in D, \ t \ge 0; \qquad (30)$$
$$u(t, x) = 0, \ x \in \partial D, \ t \ge 0;$$
$$u(0, x) = f(x), \ x \in D,$$

for $u \in \mathcal{H}_\Delta^b(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$, with the distributed order fractional derivative $\mathbb{D}^{(\rho)}$ defined by (24), is given by

$$u(t, x) = \mathbb{E}_x[f(B(E^\rho(t)))I(\tau_D(B) > E^\rho(t))]$$
$$= \mathbb{E}_x[f(B(E^\rho(t)))I(\tau_D(B(E^\rho)) > t)]$$
$$= \int_0^\infty T_D(l)f(x)g(t, l)dl$$
$$= \sum_0^\infty \bar{f}(n)\phi_n(x)h(t, \mu_n). \qquad (31)$$

In this case, $b(\lambda) = \lambda$, and $k(t)$ is given by $k(t) = Ct^{\beta_0 - 1}$, $0 < \beta_0 < 1$.

Proof. The proof is similar to the proof of Theorem 4.1 in Meerschaert et al. (2010). We give the main parts of the proof here.

Denote the Laplace transform $t \to s$ of $u(t, x)$ by

$$\tilde{u}(s, x) = \int_0^\infty e^{-st}u(t, x)dt.$$

Since we are working on a bounded domain, the Fourier transform methods in Meerschaert et al. (2002) are not useful. Instead, we will employ Hilbert space methods. Hence, given a complete orthonormal basis $\{\phi_n(x)\}$ on $L^2(D)$, we will call

$$\bar{u}(t, n) = \int_D \phi_n(x)u(t, x)dx;$$
$$\hat{u}(s, n) = \int_D \phi_n(x) \int_0^\infty e^{-st}u(t, x)dtdx$$
$$= \int_D \phi_n(x)\tilde{u}(s, x)dx$$
$$= \int_0^\infty e^{-st}\bar{u}(t, x)dt \quad \text{(when Fubini Thm holds)}$$
$$\qquad (32)$$

respectively the ϕ_n and the ϕ_n-Laplace transforms. Since $\{\phi_n\}$ is a complete orthonormal basis for $L^2(D)$, we can invert the ϕ_n-transform to obtain

$$u(t, x) = \sum_n \bar{u}(t, n)\psi_n(x)$$

for any $t > 0$, where the above series converges in the L^2 sense (e.g., see Royden (1968, Proposition 10.8.27)).

Assume that $u(t, x)$ solves (31). Using Green's second identity, we obtain

$$\int_D [u\Delta\phi_n - \phi_n\Delta u]dx = \int_{\partial D}\left[u\frac{\partial\phi_n}{\partial\theta} - \phi_n\frac{\partial u}{\partial\theta}\right]ds = 0,$$

since $u|_{\partial D} = 0 = \phi_n|_{\partial D}$, $u \in C^1(\bar{D})$ by assumption, and $\phi_n \in C^1(\bar{D})$ by Gilbarg and Trudinger (2001, Theorem 8.29). Hence, the ϕ_n-transform of Δu is

$$\int_D \phi_n(x)\Delta u(t, x)dx = -\lambda_n \int_D u(t, x)\phi_n(x)dx$$
$$= -\lambda_n\bar{u}(t, n), \qquad (33)$$

as ϕ_n is the eigenfunction of the Laplacian corresponding to eigenvalue λ_n.

The fact that the operator $\mathbb{D}^{(\rho)}$ commutes with the ϕ_n-transform follows from (28).

Taking the ϕ_n-transform of (31) we obtain that

$$\mathbb{D}^{(\rho)}\bar{u}(t, n) = -\lambda_n\bar{u}(t, n). \qquad (34)$$

From Lemma 4.2 we get the solution

$$\bar{u}(t, n) = \bar{f}(n)h(t, \lambda_n) = \bar{f}(n)\mathbb{E}(e^{-\lambda_n E^\rho(t)}).$$

Now inverting the ϕ_n-transform gives

$$u(t, x) = \sum_0^\infty \bar{f}(n)\phi_n(x)h(t, \mu_n).$$

The stochastic representation uses Lemma 4.2 and the stochastic representation of the killed semigroup of Brownian motion (2).

We use the representation (26) to establish the fact that the solution is a classical solution. The details of the proof can be seen from the proof of the main results in Meerschaert et al. (2009, 2010).

Remark 4.3. Suppose that $\mu(d\beta) = p(\beta)d\beta$, the function $\beta \mapsto \Gamma(1-\beta)p(\beta)$ is in $C^1[0, 1]$, $supp(\mu) = [\beta_0, \beta_1] \subset (0, 1)$ and $\mu(\beta_1) \ne 0$. Then

$$h(t, \lambda) = \mathbb{E}[e^{-\lambda E^\nu(t)}] = \frac{\lambda}{\pi}\int_0^\infty r^{-1}e^{-tr}\frac{\Phi_1(r)}{\Phi_2(r)}dr \qquad (35)$$

where

$$\Phi_1(r) = \int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta$$

$$\Phi_2(r) = [\int_0^1 r^\beta \cos(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta + \lambda]^2$$
$$+ [\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta]^2.$$

Suppose also that

$$C(\beta_0, \beta_1, p) = \int_{\beta_0}^{\beta_1} \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta > 0. \qquad (36)$$

Then $|\partial_t h(t, \lambda)| \leq \lambda k(t)$, where
$$k(t) = [C(\beta_0, \beta_1, p)\pi]^{-1}[\Gamma(1 - \beta_1)t^{\beta_1 - 1} + \Gamma(1 - \beta_0)t^{\beta_0 - 1}]. \tag{37}$$
In this case, $h(t, \lambda)$ is a classical solution to (23). The representation (35) is due to Kochubei (2008), which follows by inverting the Laplace transform of (23).

For $u \in \mathcal{H}_\Delta^b(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$ for k given by (37), Meerschaert et al. (2010) shows that the solution to (30), where $\mathbb{D}^{(\rho)}$ replaced with the more general $\mathbb{D}^{(\nu)}$, is a strong (classical) solution for $f \in C^1(\bar{D}) \cap C^2(D)$ for which Δf has an absolutely and uniformly convergent eigenfunction expansion w.r.t $\{\phi_n\}$. Naber (2004) studied distributed-order fractional Cauchy problem in $D = (0, M) \subset \mathbb{R}$.

Remark 4.4. The methods of this paper also apply to the Cauchy problems that are obtained by replacing Laplacian with uniformly elliptic operator in divergence form defined on C^2 functions by
$$Lu = \sum_{i,j=1}^{d} \frac{\partial (a_{ij}(x)(\partial u/\partial x_i))}{\partial x_j} \tag{38}$$
with $a_{ij}(x) = a_{ji}(x)$ and, for some $\lambda > 0$,
$$\lambda \sum_{i=1}^{n} y_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \leq \lambda^{-1} \sum_{i=1}^{n} y_i^2, \quad \forall y \in \mathbb{R}^d. \tag{39}$$
If X_t is a solution to $dX_t = \sigma(X_t)dB_t + b(X_t)dt$, $X_0 = x_0$, where σ is a $d \times d$ matrix, and B_t is a Brownian motion, then X_t is associated with the operator L with $a = \sigma\sigma^T$, see Chapters 1 and 5 in Bass (1998). Define the first exit time as $\tau_D(X) = \inf\{t \geq 0 : X_t \notin D\}$. The semigroup defined by $T_D(t)f(x) = E_x[f(X_t)I(\tau_D(X)) > t)]$ has generator L with Dirichlet boundary conditions, which follows by an application of the Itô formula.

REFERENCES

O.P. Agrawal. Solution for a fractional diffusion-wave equation defined in a bounded domain. fractional order calculus and its applications. *Nonlinear Dynam.*, 29: 145–155, 2002.

W. Arendt, C. batty, M. Hieber, and f. Neurander. *Vector-valued Laplace transforms and Cauchy problems. Monographs in Mathematics.* Birkhäuser, Basel, 2001.

B. Baeumer and M.M. Meerschaert. Stochastic solutions for fractional cauchy problems. *Fract. Calc. Appl. Anal.*, 4:481–500, 2001.

R.F. Bass. *Diffusions and Elliptic Operators.* Springer-Verlag, New York, 1998.

J. Bertoin. *Lévy processes.* Cambridge University Press, 1996.

M. Caputo. Linear models of dissipation whose q is almost frequency independent, part ii. *Geophys. J. R. Astr. Soc.*, 13:529–539, 1967.

E.B. Davies. *Spectral Theory and Differential Operators. Cambrridge Studies in Adavance Mathematics.* Cambridge University Press, Cambridge, 1995.

A. Einstein. On the theory of the brownian movement. *Annalen der Physik*, 4:371–381, 1906.

D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, New York, 2001.

R. Gorenflo and F. Mainardi. Fractional diffusion processes: Probability distribution and continuous time random walk. *Lecture Notes in Physics*, 621:148–166, 2003.

R. Gorenflo and F. Mainardi. Fractional claculus: ntegral and differential equations of fractional order. In A. Carpinteri and F. Mainardi, editors, *Fractals and Fractional Calculus in Continuum Mechanics*, pages 223–276. Springer-Verlag, Wien and New York, 1997.

A.N. Kochubei. The cauchy problem for evolution equations of fractional order. *Differential Equations*, 25:967–974, 1989.

A.N. Kochubei. Diffusion of fractional order. *Lecture Notes in Physics*, 26:485–492, 1990.

A.N. Kochubei. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.*, 340:252–281, 2008.

A.N. Kochubei. Distributed order derivatives and relaxation patterns. *J. Phys. A: Math. Theor.*, 42:315203 (9pp), 2009.

M.M. Meerschaert and H.P. Sheffler. Triangular array limits for continuous time random walks. *Stochastic Processes Appl.*, 118:1606–1633, 2008.

M.M. Meerschaert and H.P. Sheffler. Stochastic model for ultraslow diffusion. *Stochastic Processes Appl.*, 116: 1215–1235, 2006.

M.M. Meerschaert and H.P. Sheffler. Limit theorems for continuous time random walks with infinite mean waiting times. *J. Appl. Probab.*, 41:623–638, 2004.

M.M. Meerschaert, D.A. Benson, H.P. Sheffler, and B. Baeumer. Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E*, 65:1103–1106, 2002.

M.M. Meerschaert, E. Nane, and P. Vellaisamy. Fractional cauchy problems on bounded domains. *Ann. Prob.*, 37: 979–1007, 2009.

M.M. Meerschaert, E. Nane, and P. Vellaisamy. Distributed order fractional cauchy problems on bounded domains. *Unpublished, Arxiv id:0912.2521*, 2010.

R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Physics A*, 37:161–208, 2004.

K. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley, New York, 1993.

M. Naber. Distributed order fractional sub-diffusion. *Fractals*, 12:23–32, 2004.

R.R. Nigmatullin. The realization of the generalized transfer in a medium with fractal geometry. *Phys. Status Solidi B*, 133:425–430, 1986.

I. Podlubny. *Fractional Differential Equations.* Academic Press, San Diego, 1999.

H.L. Royden. *Real Analysis.* MacMillan, New York, 2nd edition, 1968.

S. Samko, A. Kilbas, and O. Marichev. *Fractional Integrals and derivatives: Theory and Applications.* Gordon and Breach, London, 1993.

K.I. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, 1999.

E. Scalas. Five years of continuous-time random walks in econophysics. In A. Namatame, editor, *Proceedings of WEHIA 2004*, pages 3–16. Kyoto, 2004.

W.R. Schneider and W. Wyss. Fractional diffusion and wave equations. *J. Math. Phys.*, 30:134–144, 1989.

G. Zaslavsky. Fractional kinetic equation for hamiltonian chaos. *Phys. D*, 76:110–122, 1994.