A generalized Gronwall inequality and its application to fractional differential equations with Hadamard derivatives ⋆

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Abstract: The Gronwall inequality, which plays a very important role in classical differential equations, is generalized to the fractional differential equations with Hadamard derivative in this paper. According to the inequality, we investigate the dependence of the solution on both the order and the initial conditions to the fractional differential equations with Hadamard derivative. Furthermore, in terms of the inequality, the estimation of the bound of the Lyapunov exponents for the Hadamard type fractional differential systems is considered.

Keywords: Generalized Gronwall inequality; Hadamard fractional derivatives; Lyapunov exponents

1. INTRODUCTION

In recent decades, although the fractional differential equations is paid more and more attentions, which just involving the Riemann-Liouville fractional calculus or the Caputo one (4) (5). The Hadamard calculus has not been mentioned so much as other kinds of fractional derivatives, even if it has been presented many years ago (2).

Firstly, some of the definitions of the Hadamard derivative and integral are presented (3).

Definition 1. The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(x)$, $\forall x > 1$, is defined by

$$ {}_H D_{1,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\ln \frac{x}{t})^{\alpha-1} f(t) \frac{dt}{t}, \qquad (1)$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2. The Hadamard derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$, of function $f(x)$ is given as follows

$$ {}_H D_{1,x}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (x\frac{d}{dx})^n \int_1^x (\ln \frac{x}{t})^{n-\alpha-1} f(t) \frac{dt}{t}. \qquad (2)$$

From the definitions above, the differences between Hadamard fractional derivative and the Riemann-Liouville fractional derivative are obvious, which at least include two aspects: firstly, no matter what the definitions of integral or derivative, the kernel in the Hadamard integral has the form of $(\ln \frac{x}{t})$ instead of the form of $(x-t)$, which is involved both in the Riemann-Liouville and Caputo integral; secondly, the Hadamard derivative has the operator $(x\frac{d}{dx})^n$, whose construction is well suited to the case of the half-axis and is invariant relative to dilation (9), while the Riemann-Liouville derivative has the operator$(\frac{d}{dx})^n$.

Secondly, some of propositions with the Hadamard calculus (derivative/integral) are formed as follows.

Proposition 1. If $0 < \alpha < 1$, the following relations hold

(i) ${}_H D_{1,x}^{-\alpha} (\ln x)^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln x)^{\beta+\alpha-1}$;

(ii) ${}_H D_{1,x}^{\alpha} (\ln x)^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln x)^{\beta-\alpha-1}$,

respectively.

Here we only prove (ii), (i) can be proved similar to (ii).

Proof. In terms of the *Definition 2*, one has

$$\begin{aligned}
& {}_H D_{1,x}^{\alpha} (\ln x)^{\beta-1} \\
&= (x\frac{d}{dx}) \cdot \frac{1}{\Gamma(1-\alpha)} \int_1^x (\ln \frac{x}{t})^{-\alpha} (\ln t)^{\beta-1} \frac{dt}{t} \\
&= (x\frac{d}{dx}) \cdot \frac{(\ln x)^{\beta-\alpha}}{\Gamma(1-\alpha)} \int_1^x (1 - \frac{\ln t}{\ln x})^{-\alpha} (\frac{\ln t}{\ln x})^{\beta-1} d\frac{\ln t}{\ln x} \\
&= (x\frac{d}{dx}) \cdot \frac{(\ln x)^{\beta-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-z)^{-\alpha} z^{(\beta-1)} dz \\
&= (x\frac{d}{dx}) \cdot \frac{(\ln x)^{\beta-\alpha}}{\Gamma(1-\alpha)} B(1-\alpha, \beta) \\
&= (x\frac{d}{dx}) \cdot \frac{(\ln x)^{\beta-\alpha}}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \cdot x \cdot \frac{d((\ln x)^{(\beta-\alpha)})}{dx}
\end{aligned}$$

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$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)} \cdot x \cdot (\beta-\alpha)(\ln x)^{(\beta-\alpha-1)}(\frac{1}{x})$$
$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln x)^{(\beta-\alpha-1)}.$$
This completes the proof.

Proposition 2. If $\alpha \geq 0$ and $\beta = 1$, for any $j = [\alpha]+1$, the following relations hold

(i) $({}_H D^\alpha_{1,t} 1)(x) = \frac{1}{\Gamma(1-\alpha)}(\ln x)^{-\alpha}$;

(ii) $({}_H D^\alpha_{1,t} (\ln t)^{\alpha-j})(x) = 0$,
in which $[\cdot]$ is the Gauss function, details can be found in (3).

Next, we will introduce the weighted space $C_{\gamma,\ln}[a,b]$, $C^n_{\delta,\gamma}[a,b]$ of the function f on the finite interval $[a,b]$, if $\gamma \in C(0 \leq Re(\gamma) < 1)$, $n-1 < \alpha \leq n$, then

$$C_{\gamma,\ln}[a,b] :=$$
$$\{f(x) : \ln(\frac{x}{a})^\gamma f(x) \in C[a,b], ||f||_{C_\gamma} = ||(\ln\frac{x}{a})^\gamma f(x)||_C \},$$
$$C_{0,\ln}[a,b] = C[a,b],$$
and
$$C^n_{\delta,\gamma}[a,b] := \{g(x) : (\delta^n g)(x) \in C_{\gamma,\ln}[a,b],$$
$$||g||_{C_{\gamma,\ln}} = \sum_{k=0}^{n-1} ||\delta^k g||_C + ||\delta^n g||_{C_{\gamma,\ln}}\},$$
$$\delta = x\frac{d}{dx}.$$

Theorem 1. Let $\alpha > 0$, $n = -[-\alpha]$ and $0 \leq \gamma < 1$. Let G be an open set in R and let $f : (a,b] \times G \longrightarrow R$ be a function such that: $f[x,y] \in C_{\gamma,\ln}[a,b]$ for any $y \in G$, then the following problem
$$ {}_H D^\alpha_{a,t}(x) = f[x,y(x)], (\alpha > 0), \tag{3}$$
$$ {}_H D^{\alpha-k}_{a,t}(a+) = b_k, b_k \in R, (k = 1,\cdots,n, n = -[-\alpha]), \tag{4}$$
satisfies the following Volterral integral equation:
$$y(x) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha-j+1)}(\ln\frac{t}{a})^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_a^x (\ln\frac{x}{t})^{\alpha-1} f[t,y(t)]\frac{dt}{t}, (x > a > 0), \tag{5}$$
i.e, $y(x) \in C_{n-\alpha,\ln}[a,b]$ satisfies the relations 3-4 if and only if it satisfies the Volterra integral equation 5.

In particular, if $0 < \alpha \leq 1$, the problem 3-4 is equivalent to the following equation:
$$y(x) = \frac{b}{\Gamma(\alpha)}(\ln\frac{t}{a})^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^x (\ln\frac{x}{t})^{\alpha-1} f[t,y(t)]\frac{dt}{t}, (x > a > 0). \tag{6}$$
Details can be found in (3).

2. THE GENERALIZED GRONWALL INEQUALITY

The Gronwall inequality, which plays a very important part in classical differential systems, has been generalized by Ye et al., recently, which can be used in fractional differential equations with Riemann-Liouville derivatives (10). The inequality plays a useful role in fractional differential equations, such as the dependence of the solution on the order, and the initial conditions for Riemann-Liouville fractional differential systems.

This paper would present a generalized Gronwall inequality which has a close connection to the Hadamard derivative.

Firstly, let's present the classical Gronwall inequality which can be found in (1).

Theorem 2. If
$$x(t) \leq h(t) + \int_{t_0}^t k(s)x(s)ds, t \in [t_0,T),$$
where all the functions involved are continuous on $[t_0,T)$, $T \leq \infty$, and $k(t) \geq 0$, then $x(t)$ satisfies
$$x(t) \leq h(t) + \int_{t_0}^t h(s)k(s)exp[\int_s^t k(u)du]ds, t \in [t_0,T).$$
If, in addition, $h(t)$ is nondecreasing, then
$$x(t) \leq h(t)exp(\int_{t_0}^t k(s)ds), t \in [t_0,T).$$

Secondly, the generalized Gronwall inequality with Riemann-Liouville fractional derivatives is presented as follows (10).

Theorem 3. Suppose $\alpha > 0$, $a(t)$ is a nonnegative function and locally integrable on $0 \leq t < T$ (some $T \leq +\infty$) and $g(t)$ is a nonnegative, nondecreasing, continuous function defined on $0 \leq t < T$, $g(t) \leq M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with
$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)ds,$$
on the interval. Then
$$u(t) \leq a(t) + \int_0^t [\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)]ds, \quad 0 \leq t < T.$$

This inequality has a close connection to the Riemann-Liouville derivative. It also can be used to estimate the bound of the Lyapunov exponents for both the Riemann-Liouville fractional differential systems and the Caputo ones. In the following, we will give a similar and different inequality which has a great application to the Hadamard derivative.

Theorem 4. Suppose $\alpha > 0$, $a(t)$ and $u(t)$ are nonnegative functions and locally integrable on $1 \leq t < T$ (some $T \leq +\infty$), and $g(t)$ is a nonnegative, nondecreasing, continuous function defined on $1 \leq t < T$, $g(t) \leq M$ (constant). If the following inequality
$$u(t) \leq a(t) + g(t)\int_1^t (\ln\frac{t}{s})^{\alpha-1}u(s)\frac{ds}{s}, \ 1 \leq t < T, \tag{7}$$
holds. Then
$$u(t) \leq a(t) + \int_1^t [\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\ln\frac{t}{s})^{n\alpha-1}a(s)]\frac{ds}{s}, \quad 1 \leq t < T. \tag{8}$$

Proof. Let $B\phi(t) = g(t)\int_1^t (\ln\frac{t}{s})^{n\alpha-1}\phi(s)\frac{ds}{s}$, $t \geq 1$, for locally integrable functions ϕ. Then

$$u(t) \leq a(t) + Bu(t).$$

Iterating the inequality, one has

$$u(t) \leq \sum_{k=0}^{n-1} B^k a(t) + B^n u(t).$$

In the following, we should prove that

$$B^n u(t) \leq \int_1^t \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} (\ln\frac{t}{s})^{n\alpha-1} u(s)\frac{ds}{s}, \qquad (9)$$

holds, and $B^n u(t) \to +\infty$ for each t in $1 \leq t < T$.

Obviously, the relation 9 holds as $n = 1$. Suppose it holds for some $n = k$. If $n = k+1$, then one has

$$\begin{aligned} &B^{k+1}u(t) = B(B^k u(t)) \\ &\leq g(t)\int_1^t (\ln\frac{t}{s})^{\alpha-1}[\int_1^s \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\ln\frac{s}{\tau})^{k\alpha-1}u(\tau)\frac{d\tau}{\tau}]\frac{ds}{s}. \end{aligned}$$

Under the condition that $g(t)$ is nondecreasing, one obtains

$$\begin{aligned} &B^{k+1}u(t) \\ &\leq (g(t))^{k+1}\int_1^t (\ln\frac{t}{s})^{\alpha-1}[\int_1^s \frac{\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\ln\frac{s}{\tau})^{k\alpha-1}u(\tau)\frac{d\tau}{\tau}]\frac{ds}{s}. \end{aligned}$$

By interchanging the order of integration, one has

$$\begin{aligned} &B^{k+1}u(t) \\ &\leq (g(t))^{k+1}\int_1^t [\int_\tau^t \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)}(\ln\frac{t}{s})^{\alpha-1}(\ln\frac{s}{\tau})^{k\alpha-1}\frac{ds}{s}]u(\tau)\frac{d\tau}{\tau} \\ &= \int_1^t \frac{(g(t)\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)}(\ln\frac{t}{s})^{(k+1)\alpha-1}u(s)\frac{ds}{s}, \end{aligned}$$

where the integral

$$\begin{aligned} &\int_\tau^t \ln\frac{t}{s})^{\alpha-1}(\ln\frac{s}{\tau})^{k\alpha-1}\frac{ds}{s} \\ &= (\ln\frac{t}{\tau})^{k\alpha+\alpha-1}\int_0^1 (1-z)^{\alpha-1}z^{k\alpha-1}dz \\ &= (\ln\frac{t}{\tau})^{(k+1)\alpha-1}B(k\alpha,\alpha) \\ &= \frac{\Gamma(\alpha)\Gamma(k\alpha)}{\Gamma((k+1)\alpha)}(\ln\frac{t}{\tau})^{(k+1)\alpha-1}, \end{aligned}$$

is obtained in terms of the condition that $\ln s = \ln\tau + z\ln\frac{t}{\tau}$.

Therefore, the relation 9 holds.

Moreover, since

$$B^n u(t) \leq \int_1^t \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\ln\frac{t}{s})^{n\alpha-1}u(s)\frac{ds}{s} \to 0,$$

as $n \to +\infty$, for $t \in [1,T)$.

This completes the proof.

Corollary 1. Let $g(t) = b$ in relation 7. Here b is a positive constant. The relation 7 turns into the following form

$$u(t) \leq a(t) + b\int_1^t (\ln\frac{t}{s})^{\alpha-1}u(s)\frac{ds}{s}.$$

According to the *Theorem 4*, we will obtain

$$\begin{aligned} &u(t) \leq \\ &a(t) + \int_1^t [\sum_{n=1}^{\infty}\frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\ln\frac{t}{s})^{n\alpha-1}a(s)]\frac{ds}{s},\ (1 \leq t < T). \end{aligned}$$

Corollary 2. Under the assumption of *Theorem 4*, we restrict $a(t)$ to be a nondecreasing function on $[1,T)$. Then

$$u(t) \leq a(t)E_{\alpha,1}(g(t)\Gamma(\alpha)(\ln t)^\alpha),$$

where $E_{\alpha,1}$ is the Mittag-Leffler function defined by

$$E_{\alpha,1} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}$$

.

Proof. The assumptions imply

$$\begin{aligned} &u(t) \\ &\leq a(t)[1 + \int_1^t \sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(\ln\frac{t}{s})^{n\alpha-1}\frac{ds}{s}] \\ &= a(t)\sum_{n=0}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha+1)} \\ &= a(t)E_\alpha(g(t)\Gamma(\alpha)(\ln t)^\alpha). \end{aligned}$$

This completes the proof.

3. THE DEPENDENCE OF SOLUTION ON PARAMETERS

As far as we are concerned, there have been several papers dedicated to study the dependence of the solution on the order and the initial condition to the fractional differential equation with Riemann-Liouville type or Caputo type derivatives, while quite few papers are committed to study the same problems with Hadamard type derivatives. In this section, we will display how the theorem which we get makes contribution to analyzing the dependence of the solution on the order and the initial condition to the fractional differential equation with Hadamard fractional derivatives.

Now we consider the following fractional system in terms of the Hadamard fractional derivatives:

$${}_HD_{1,t}^{\alpha}y(t) = f(t, y(t)), \qquad (10)$$

$${}_HD_{1,t}^{\alpha-1}y(t)|_{t=1} = \eta, \qquad (11)$$

where $0 < \alpha < 1, 1 \leq t < T \leq +\infty, f : [1,T) \times R \to R$.

The existence and uniqueness of the initial value problem 10-11 have been studied in (3), in which one also discussed the dependence of a solution on initial conditions. Here, we would investigate the dependence on both initial value and order under the generalized inequality with the Hadamard fractional derivatives.

First, let reduce the problem 10-11 to the Volterra integral equation.

$$y(t) = \frac{\eta}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\ln t)^{\alpha-1}f(\tau, y(\tau))\frac{d\tau}{\tau}. \qquad (12)$$

Obviously, the Volterra equation is equivalent to the initial value problem 10-11.

Theorem 5. Let $\alpha > 0$ and $\delta > 0$ such that $0 < \alpha - \delta < \alpha \leq 1$. Also let the function f be continuous and satisfy the Lipschitz condition with respect to the second variable:

$$|f(t,y)-f(t,z)|\leq L|y-z|$$

for a constant L independent of t,y,z in R. For $1\leq t\leq h<T$, assume that y and z are the solutions of the initial value problems 10-11 and

$$_{H}D_{1,t}^{\alpha-\delta}z(t)=f(t,z(t)), \tag{13}$$

$$_{H}D_{1,t}^{\alpha-\delta-1}z(t)|_{t=1}=\bar{\eta}, \tag{14}$$

respectively. Then, the following relation holds for $1<t\leq h$:

$$\begin{aligned}&|z(t)-y(t)|\\ &\leq A(t)+\int_1^t[\sum_{n=1}^{\infty}(\frac{L}{\Gamma(\alpha)}\Gamma(\alpha-\delta))^n\frac{(\ln\frac{t}{s})^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))}A(s)]\frac{ds}{s},\end{aligned}$$

where

$$\begin{aligned}&A(t)\\ &=|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}(\ln t)^{\alpha-\delta-1}-\frac{\eta}{\Gamma(\alpha)}(\ln t)^{\alpha-1}|\\ &+|\frac{(\ln t)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)}-\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}|\cdot\|f\|\\ &+|\frac{(\ln t)^{\alpha-\delta}}{\alpha-\delta}[\frac{1}{\Gamma(\alpha-\delta)}-\frac{1}{\Gamma(\alpha)}]|\cdot\|f\|,\end{aligned}$$

and

$$\|f\|=\max_{1\leq t\leq h}|f(t,y)|.$$

Proof. The solutions of the initial value problem 10-11 and 13-14 are as follows:

$$y(t)=\frac{\eta}{\Gamma(\alpha)}(\ln t)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t)^{\alpha-1}f(\tau,y(\tau))\frac{d\tau}{\tau},$$

and

$$\begin{aligned}z(t)\ =\ &\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}(\ln t)^{\alpha-\delta-1}\\ &+\frac{1}{\Gamma(\alpha-\delta)}\int_1^t(\ln t)^{\alpha-\delta-1}f(\tau,z(\tau))\frac{d\tau}{\tau}.\end{aligned}$$

So we have

$$\begin{aligned}&|z(t)-y(t)|\\ \leq\ &|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}(\ln t)^{\alpha-\delta-1}\\ &-\frac{\eta}{\Gamma(\alpha)}(\ln t)^{\alpha-1}|\\ &+|\frac{1}{\Gamma(\alpha-\delta)}\int_1^t(\ln t)^{\alpha-\delta-1}f(\tau,z(\tau))\frac{d\tau}{\tau}\\ &-\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t)^{\alpha-\delta-1}f(\tau,z(\tau))\frac{d\tau}{\tau}|\\ &+|\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t)^{\alpha-\delta-1}f(\tau,z(\tau))\frac{d\tau}{\tau}\\ &-\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t)^{\alpha-\delta-1}f(\tau,y(\tau))\frac{d\tau}{\tau}|\\ &+|\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t)^{\alpha-\delta-1}f(\tau,y(\tau))\frac{d\tau}{\tau}\\ &-\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t)^{\alpha-1}f(\tau,y(\tau))\frac{d\tau}{\tau}|\\ \leq\ &A(t)+\frac{1}{\Gamma(\alpha)}\int_1^t(\ln t)^{\alpha-\delta-1}L|z(\tau)-y(\tau)|\frac{d\tau}{\tau},\end{aligned}$$

where

$$\begin{aligned}A(t)\ =\ &|\frac{\bar{\eta}}{\Gamma(\alpha-\delta)}(\ln t)^{\alpha-\delta-1}-\frac{\eta}{\Gamma(\alpha)}(\ln t)^{\alpha-1}|\\ &+|\frac{(\ln t)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)}-\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}|\cdot\|f\|\\ &+|\frac{(\ln t)^{\alpha-\delta}}{\alpha-\delta}[\frac{1}{\Gamma(\alpha-\delta)}-\frac{1}{\Gamma(\alpha)}]|\cdot\|f\|.\end{aligned}$$

Applying *Theorem 1* to the above inequality and yields

$$\begin{aligned}&|z(t)-y(t)|\\ &\leq A(t)+\int_1^t[\sum_{n=1}^{\infty}(\frac{L}{\Gamma(\alpha)}\Gamma(\alpha-\delta))^n\frac{(\ln\frac{t}{s})^{n(\alpha-\delta)-1}}{\Gamma(n(\alpha-\delta))}A(s)]\frac{ds}{s}.\end{aligned}$$

This completes the proof.

It follows from *Theorem 5* that for every small changes $\varepsilon\in[1,h]$ in order and initial condition cause only small changes of the solution in the closed interval $[\varepsilon,h]$.

Next, we will give a example to discuss the approximate solution of the Hadamard fractional differential equation.

$$_{H}D_{1,t}^{1-\delta}x(t)=x(t), \tag{15}$$

$$_{H}D_{1,t}^{-\delta}x(t)|_{t=1}=1, \tag{16}$$

where $1\leq t<T\leq+\infty$, δ is a small positive constant.

For the question above, we don't necessarily bother to get its asymptotic solution. We can find its approximate solution quickly in other way. Now we consider the simple problem as follows.

$$_{H}D_{1,t}^{1}y(t)=y(t), \tag{17}$$

$$_{H}D_{1,t}^{0}y(t)|_{t=1}=1. \tag{18}$$

Combining the corresponding evaluation and the *Theorem 5*, one has

$$\begin{aligned}&A(t)=\\ &|\frac{1}{\Gamma(1-\delta)}(\ln t)^{-\delta}-1|+|\frac{(\ln t)^{1-\delta}}{1-\delta}-\ln t|\cdot\|x\|\\ &+|\frac{(\ln t)^{1-\delta}}{1-\delta}[\frac{1}{\Gamma(1-\delta)}-1]|\cdot\|y\|.\end{aligned}$$

When $\delta\longrightarrow 0$ and $t\in[1,T)$, we get $A(t)\longrightarrow 0$.

Actually, $\delta\longrightarrow 0$ and $t\in[1,T)$, one has

$$|x(t)-y(t)|=|e^{\ln t}-(\ln t)^{\delta}e^{(\ln t)^{1-\delta}}|\longrightarrow 0.$$

Therefore, when δ is a small parameter, we can reasonably substitute $x(t)=e^{\ln t}$ for the solution of the problem 17-18. The example shows that the Hadamard differential equation has the dependence on both initial conditions and the order of derivative.

4. ESTIMATION OF THE BOUND OF THE LYAPUNOV EXPONENTS FOR THE FRACTIONAL DIFFERENTIAL SYSTEMS WITH HADAMARD DERIVATIVE

Theorem 6. The following fractional differential system with Hadamard derivative

$$\begin{cases} {}_HD^{\alpha}_{t_0,t}x(t) = f(x,t), \\ (x,t) \in \Omega \times (t_0,+\infty) \subset \mathbb{R}^n \times (t_0,+\infty),\ \alpha \in (0,1), t_0 > 0, \\ {}_HD^{\alpha-1}_{t_0,t}x(t)|_{t=t_0} = x_0, \end{cases} \tag{19}$$

has its first variation equation

$$\begin{cases} {}_HD^{\alpha}_{t_0,t}\Phi(t) = f_x(x,t)\Phi(t), \\ (x,t) \in \Omega \times (t_0,+\infty) \subset \mathbb{R}^n \times (t_0,+\infty),\ \ \alpha \in (0,1), t_0 > 0, \\ \Phi(t_0) = I, \end{cases} \tag{20}$$

where I is a identity and

$$\Phi(t) = \frac{\partial}{\partial s}\phi(t; x_0 + s\Phi(t))|_{s=0} = D_x\phi(t_0; x_0),$$

$\phi(t_0; x_0)$ is the fundamental solution to the system.

Proof. The proof is similar to the reference (5), we omit the details here.

Definition 3. Let $u_k(t),\ k = 1, 2, \cdots, n$ be the eigenvalues of $\Phi(t)$ of system 20, which satisfy $|u_1(t)| \le |u_2(t)| \le \cdots \le |u_n(t)|$. Then the Lyapunov exponents l_k of the trajectory $x(t)$ solving 20 are defined by

$$l_k = \lim_{t\to\infty} \sup \frac{1}{t} \ln |u_k(t)|, \quad k = 1, 2, \cdots, n.$$

These exponents l_k, $k = 1, 2, \cdots, n$, are real numbers. The existence of the limit for the classical differential system was established (7). For the fractional differential system, it still holds. Obviously, Φ is not invertible when $u_1(t) = 0$, which implies $l_1 = -\infty$. But this case does not happen in general. Hence, here and hereafter, we assume that $u_1(t)$ is not (identically) equal to zero. Therefore, Φ is always supposed to be invertible.

Next, we will get down to estimate the bound of the Lyapunov exponents for the fractional differential systems with Hadamard derivative. But firstly, let's take a look at the following lemma.

Lemma 1. If $0 < \alpha < 2$, β is an arbitrary complex number, u is an arbitrary real number such that $\frac{\pi\alpha}{2} < u < \min\{\pi, \pi\alpha\}$, then for an arbitrary integer $p \ge 1$ the following expansion holds

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O(|z|^{-1-p}),$$

$|z| \to \infty,\ |\arg(z)| \le u.$

The lemma can be found in (8).

By *Lemma 1*, we can directly obtain the asymptotic expansion of the Mittag-Leffler function

$$E_{\alpha,\alpha}(K(\ln t)^{\alpha}) \approx \frac{e^{K^{\frac{1}{\alpha}}}}{\alpha} K^{\frac{1}{\alpha}-1}(\ln t)^{1-\alpha} t, \quad t \to +\infty,$$

where K is a positive constant.

Integrating system 19 gives

$$\Phi(t) = \frac{(\ln \frac{t}{t_0})^{\alpha-1}}{\Gamma(\alpha)} I + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (\ln \frac{t}{\tau})^{\alpha-1} f_x(x,\tau)\Phi(\tau) \frac{d\tau}{\tau}.$$

Taking the matrix norm of both sides of the above equation leads to

$$\|\Phi(t)\| \le \frac{(\ln \frac{t}{t_0})^{\alpha-1}}{\Gamma(\alpha)} + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t} (\ln \frac{t}{\tau})^{\alpha-1} \|\Phi(\tau)\| \frac{d\tau}{\tau},$$

where we assume that the constant $M \ge \|f_x(x,t)\|$.

Applying *Corollary 2* to the above integral inequality brings about

$$\|\Phi(t)\| \le (\ln \frac{t}{t_0})^{\alpha-1} E_{\alpha,\alpha}(M(\ln \frac{t}{t_0})^{\alpha}).$$

By the fact that the spectral radius of a given matrix is not bigger than any norm of a matrix, we have

$$|u_n(t)| \le \|\Phi(t)\| \le (\ln \frac{t}{t_0})^{\alpha-1} E_{\alpha,\alpha}(M(\ln \frac{t}{t_0})^{\alpha}).$$

By the definition of the Lyapunov exponents and applying *Lemma 1*, one gets

$$\begin{aligned} l_n &= \lim_{t\to+\infty} \sup \frac{1}{t} \ln |u_n(t)| \le \lim_{t\to+\infty} \sup \frac{1}{t} \ln \|\Phi(t)\| \\ &\le \lim_{t\to+\infty} \sup \frac{1}{t} \ln((\ln \frac{t}{t_0})^{\alpha-1} E_{\alpha,\alpha}(M(\ln \frac{t}{t_0})^{\alpha})) \\ &= \lim_{t\to+\infty} \sup \frac{1}{t} \ln(\frac{e^{K^{\frac{1}{\alpha}}}}{\alpha} K^{\frac{1}{\alpha}-1} (\ln \frac{t}{t_0})^{1-\alpha} \frac{t}{t_0}) \\ &= 0. \end{aligned}$$

Theorem 7. The Lyapunov exponents of systems 19 satisfy

$$-\infty < l_1 \le \cdots \le l_n \le 0,$$

where $\|f_x(x,t)\| \le M$.

So we eventually derive the upper bound of the Lyapunov exponents for the fractional differential systems with Hadamard derivatives and the exact upper bound is zero, which means that generally there are no chaotic attractors about the fractional differential systems with Hadamard derivative in terms of *Definition 3*.

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