The use of fractional order models in predictive control

Djalil BOUDJEHEM * ** **Badreddine BOUDJEHEM** * ***

* *Laboratory of Automatics and Informatics of Guelma L.A.I.G*
** *Department of Electrical Engineering University of Guelma BP 401, Guelma 24000, Algeria (e-mail: dj_boudjehem@yahoo.fr)*
*** *Department of Electrotechnique, University of Skikda (e-mail: b_boudjehem@yahoo.fr)*

Abstract: Fractional order dynamic model could model various real materials more adequately than integer order ones and provide a more adequate description of many actual dynamical processes. In this paper, we present the use of fractional order system representation in the Model Predictive Control (MPC) to describe the dynamics of plant used to construct the control law. The use of the fractional model guarantees stability and performance of the closed-loop especially with the present of noise. Simulation results are presented to show that the use of fractional order MPC achieves better control performance compared to those of the conventional MPC that uses integer order models.

Keywords: Fractional calculus, fractional systems, fraction order approximation, predictive control, receding horizon.

1. INTRODUCTION

Fractional calculus allows a more compact representation and problem solution for many systems. The idea of fractional integrals and derivatives has been known since the development of regular calculus. Probably the first physical system to be widely recognized as one demonstrating fractional behavior is the semi-infinite lossy (RC) transmission line (see Clarke et al. (2004)). Another equivalent system is the diffusion of heat into semi-infinite solid (see Kulish and Lage (2000)). Other systems that are known to display fractional order dynamics are viscoelasticity, colored noise, electrode-electrolyte polarization, dielectric polarization, boundary layer effects in ducts, and electromagnetic waves. Because many systems are known to display fractional order dynamics, they can't be controlled the same way as those which doesn't. Unfortunately, these systems had been considered to be similar to systems with integer order dynamics for a long time. However, in the last decade we notice the born of the fractional control that deals with those specific systems. The significance of fractional control system is that it is a generalization of the classical integer order control theory, which could lead to a more adequate modeling and more robust control performance.

Predictive control is a family of control techniques that optimize a given criterion by using a model to predict system evolution and compute a sequence of future control actions. Predictive control accepts a variety of models, objective functions, and constraints, providing flexibility in handling a wide range of operating criteria present in industrial processes. See Maciejowski (2002), Camacho and Bordóns (2004) and Rossiter (2003). A variety of processes can be controlled using MPC. Guzmán et al. (2005)
Generally, in MPC linear models are used to predict the system dynamics, even though the dynamics of the closed-loop system is nonlinear or displays fractional order dynamics.

This paper focuses on the use of fractional order system description to model fractional order dynamics in model predictive control to construct a fractional order Model Predictive Control. Where the fractional order model could model various real materials more adequately than integer order ones and provide a more adequate description of many actual dynamical processes, which will improve the MPC performances and lead to a more robust control performance. See Valério and da Costa (2006) and Shantanu (2008). The results in this paper show that the use of fractional order models don't only give better results than the use of integer order models, but it performs better with the presence of noise which, can be interpreted as a kind of robustness.

The paper is organized as follows. In Section 1 we present some theoretical aspects of fractional order systems and fractional order approximation. In Section 2 . the basic concept of MPC is introduced. An outline of the fractional model MPC schemes is also presented and simulation results are discussed in section 3.

2. FRACTIONAL ORDER SYSTEMS

A fractional order system is that system described by the following fractional order differential equation

$$a_n D^{\alpha_n} f(x) + a_{n-1} D^{\alpha_{n-1}} f(x) + a_{n-2} D^{\alpha_{n-2}} f(x) + \ldots$$
$$= b_n D^{\beta_n} f(x) + b_{n-1} D^{\beta_{n-1}} f(x) + b_{n-2} D^{\beta_{n-2}} f(x) + \ldots \quad (1)$$

where $D^{\alpha_n} = _0 D_t^{\alpha_n}$, is called the fractional derivative of order α_n with respect to variable t and with the starting point $t = 0$.
In fractional calculus, the fractional derivative is defined due to Riemann and Liouville fractional integral version given by (2). (See Miller and Ross (1993) and Podlubny (1999a)):

$$_0D_x^{-\nu} f(x) = \int_0^x \frac{(x-t)^{\nu-1}}{(\nu-1)!} f(t)dt. \quad (2)$$

Or

$$_0D_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt. \quad (3)$$

where $\Gamma(\nu)$ is the Euler's *Gamma* function defined by: $\Gamma(\nu) = (\nu - 1)!$, with the property: $\Gamma(\nu + 1) = \nu\Gamma(\nu)$.

The fractional derivation is then defined by:

$$D^\nu f(x) = \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\nu-1} f(t)dt. \quad (4)$$

where n is an integer number defined as $n - 1 < \nu < n$

The Laplace transform of a fractional derivative can be calculated easily by applying the regular Laplace operator on (3):

$$\mathcal{L}[D^\nu f(x)] = s^\nu F(s) - \sum_{k=0}^{n-1} s^{n-k-1}\ D^{k-n+\nu} f(0) \text{ if } Re(\nu) > 0 \quad (5)$$

and n is the integer number defined before. Where the summation in the right hand side of (5) will be equals to zero if $Re(\nu) \leq 0$

Approximation of the fractional order transfer function

For simulations and implementations, we need to approximate the fractional order transfer functions of powers of $\nu \in \mathbb{R}$ by the usual integer order $n \in \mathbb{Z}$ transfer functions with a similar behavior. The integer transfer function may then have to include an infinite number of poles and zeroes. But it is always possible to get good approximations. In fact, there are a number of approximations that exist and makes use of a recursive distribution of poles and zeroes. See Oustaloup (1991) and Chareff et al. (1992) . The most common approximation used is that proposed by Oustaloup (1991):

$$s^\nu \approx C \prod_{k=-N}^{N} \frac{1 + (s/\omega_{z_k})}{1 + (s/\omega_{p_k})}, \quad \nu > 0. \quad (6)$$

ω_l, ω_h are the lower and higher frequency approximation interval. This means that the approximation is valid in that frequency interval. The gain C has the role of approximation tuning, so it is adjusted until both sides of (6) will have unit gain at 1 rad/s. The approximation limits N is chosen before hand, and the good performance of the approximation strongly depends thereon. low values result in simpler approximations, but also cause the appearance of a ripple in both gain and phase behaviors; such a ripple may be practically eliminated increasing N, but the approximation will be computationally heavier. Frequencies of poles and zeroes in (6) are given by:

$$\alpha = (\omega_h/\omega_l)^{\nu/N} \quad (7)$$
$$\eta = (\omega_h/\omega_l)^{(1-\nu)/N} \quad (8)$$
$$\omega_{z_0} = \omega_0\sqrt{\eta} \quad (9)$$
$$\omega_{p_k} = \omega_{z_k}\alpha \quad (10)$$
$$\omega_{z_{k+1}} = \omega_{p_k}\eta \quad (11)$$

In general, it is usual to split fractional powers of s like this:

$$s^\nu = s^n s^\delta, \quad \nu = n + \delta \quad (12)$$

where n is an integer number defined as: $n < \nu < n + 1$, thus the values of δ will be compromise between 0 and 1. In this manner we need only to approximate the latter term.

3. THE MODEL PREDICTIVE CONTROL

Model Predictive Control problem is formulated as solving on-line a finite horizon open-loop optimal control problem subject to system dynamics and constraints involving states and controls. However, the success of the model predictive control strategy depends critically on the choice of the model. See Maciejowski (2002) and Camacho and Bordóns (2004) and Guzmán et al. (2005).

3.1 The model predictive control principle

For a given plant, at instant k, the reference trajectory $r(k)$ is defined to be the ideal trajectory along which the plant should return to the set-point trajectory $w(k)$. It is frequently assumed that the reference trajectory approaches the set-point exponentially from the current output value $y(k)$, with the time constant of the exponential, which we shall denote T_{ref}, defining the speed of response [Camacho and Bordóns (2004)].
The current error between the output and the setpoint is then defined to be:

$$err(k) = w(k) - y(k) \quad (13)$$

then the reference trajectory is chosen such that the error i steps later, if the output followed it exactly, would be

$$err(k+i) = \exp^{-iT_s/T_{ref}} err(k) \quad (14)$$

where T_s is the sampling interval. That is the reference trajectory is defined to be:

$$r(k+i|k) = w(k+i) - err(k+i) \quad (15)$$

The notation $r(k+i|k)$ indicates that the reference trajectory depends on the conditions at time k, in general. See Maciejowski (2002) and Richalet (1993).
A predictive controller has an internal model which is used to predict the behavior of the plant, starting at the current time k, over a future prediction horizon H_p. This predicted behavior depends on the assumed input trajectory $\hat{u}$ that

is to be applied over the prediction horizon, and the idea is to select that input which promises the best predicted behavior. The notation $\hat{u}$ rather than u here indicates that at time k we only have a prediction of what the input at time $k+i$ may be; the actual input at that time, $u(k+i)$, will probably be different from $\hat{u}(k+i|k)$. Note that we assume that we have the output measurement $y(k)$ available when deciding the value of the input $u(k)$. We should also notice that the model output $y(k)$ depends only on the past inputs $u(k-1), u(k-2), ...$, not including the present input $u(k)$. Maciejowski (2002).

Once a future input trajectory has been chosen, only the first element of that trajectory is applied as the input signal to the plant. That is, we set $u(k) = \hat{u}(k|k)$, where $u(k)$ denotes the actual signal applied. Then the whole cycle of output measurement, prediction, and input trajectory determination is repeated, one sampling interval later. Since the prediction horizon remains of the same length as before, but slides along by one sampling interval at each step, this way of controlling a plant is often called a receding horizon strategy.

3.2 Computing the optimal control signal

To compute the control signal $\hat{u}(k)$ that will be applied at time instant k, we should solve an optimization problem where, the receding horizon cost function that will be minimized is defined by 16.

$$J = \sum_{i \in P} R \times [r(k+i|k) - y(k+i|k)]^2 + \sum_{H_u} Q \times [\Delta u(k+i-1)]^2 \tag{16}$$

where P denotes the set of indices i which correspond to coincidence points and H_u is the control horizon, R and Q are weighting matrices. In the simplest case these matrices are set to identity matrix.

Conceptually, the internal model can first be used to predict the free responses $\hat{y}_f(i + H_p|k)$ of the plant, which are the responses that would be obtained at the coincidence points if the future input trajectory remained at the latest value $u(k-1)$. Depending on the form of the model, these values will be obtained, and if a step or pulse response is available as the model, then all the available past inputs are needed.

Now let $S(H_p)$ be the response of the model to a unit step input, H_p steps after the unit step is applied. The predicted output at time $k + H_p$ is

$$\hat{y}(k+Hp|k) = \hat{y}_f(k+Hp|k) + S(H_p)\Delta\hat{u}(k|k) \tag{17}$$

where

$$\Delta\hat{u}(k|k) = \hat{u}(k|k) - u(k-1) \tag{18}$$

is the change from the current input $u(k-1)$ to the predicted input $\hat{u}(k|k)$. We want to achieve

$$\hat{y}(k+Hp|k) = r(k+Hp|k) \tag{19}$$

hence, the optimal change of input can be calculated easily by solving the minimization of the cost function given by (16).

4. FRACTIONAL MODEL FOR PREDICTIVE CONTROL

Since fractional order models describes fractional systems better than integer order models do, we propose to use a fractional order model rather than an integer order one as in the used conventional model predictive control.

Figure 1 shows the model predictive control scheme using

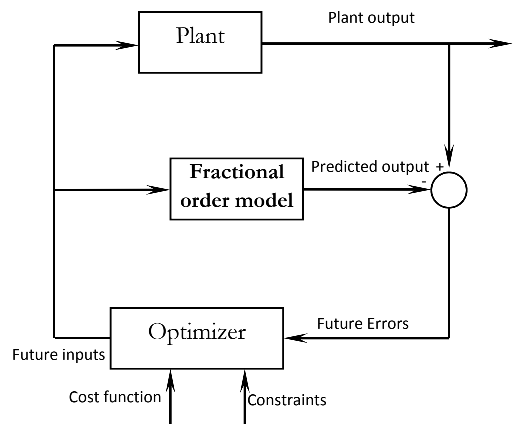

Fig. 1. Model predictive control using fractional order model

fractional order model. The implementation of this idea means that (16) have to be in the following forme:

$$J = \sum_{i \in P} [r(k+i) - y_{frac}(k+i|k)]^2 + \sum_{H_u} [\Delta u(k+i-1)]^2 \tag{20}$$

where y_{frac} is the fractional order system output. Due to the approximation calculation of y_{frac} the determination of the control signal $u(k)$ or its increment $\Delta u(k)$ will be quit difficult, and for this reason the matrices R and Q are set to identity. Therefor, depending on the number of poles and zeros used for the fractional system approximation, we can adjust the control system for better improvement. Consequently, a better fractional system dynamic approximation will be achieved.

We will show in the next section how can we improve the performances of the predictive control, and maintain the system stability in the case of changing the prediction horizon by the use of a fractional order model rather than integer order model in the predictive control.

4.1 Simulation results

In this section we consider the following non commensurate fractional order plant given by Podlubny (See Podlubny (1999a)):

$$G(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} \tag{21}$$

To implement the fractional system model we used the approximation given by (6) and (7) to (11).

For the comparison purpose, we will use the integer order model proposed by Podlubny (See Podlubny (1999b)) given by (22) , which represent the nearest integer order model to the system defined by 21. Since the fractional orders are approximated to those nearest integer ones This model will be used to implement the integer order MPC:

$$G(s) = \frac{1}{0.8s^2 + 0.5s + 1} \tag{22}$$

The performance of the results is then asserted and compared to the performance obtained with integer order model.
Figure 2 shows the step response of the fractional order system.
Figures 3 and 4 shows the effect of the prediction horizon

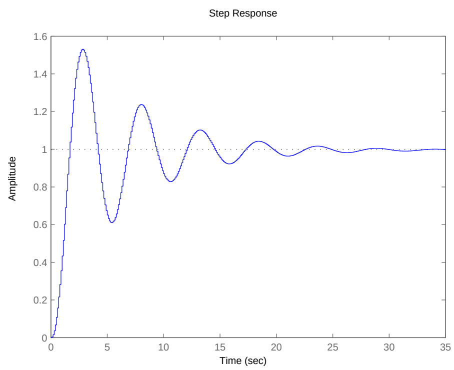

Fig. 2. Step response of the plant

on the closed loop control system using both fractional order and integer order MPC respectively. In figure 3 the step responses shows an increase in the maximum overshoot as the prediction horizon increases. Unfortunately, this not the case for the integer order MPC as shown in figure 4. Since integer order MPC was completely unstable for a prediction horizon $H_p = 6$ the prediction horizon started from $H_p = 8$.
Hence, for the comparison reasons we used the prediction horizon $H_p = 12$ with both MPC's.

Figure 6 shows the controlled plant output to a square input signal using both fractional order and conventional MPC. Notice that both plant outputs reaches the correct set-point. However, the use of the fractional order MPC leads to better improvements of the control of the fractional order system compared to the use of the integer order MPC.
Figure 6 shows the control input signal using both frac-

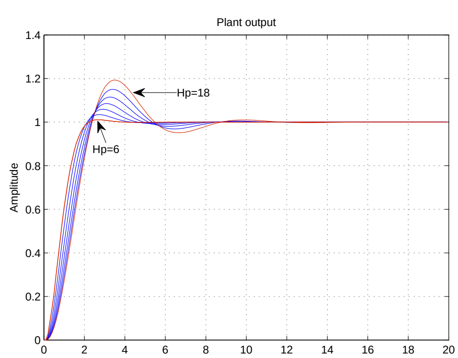

Fig. 3. The effect of the prediction horizon on the fractional model MPC

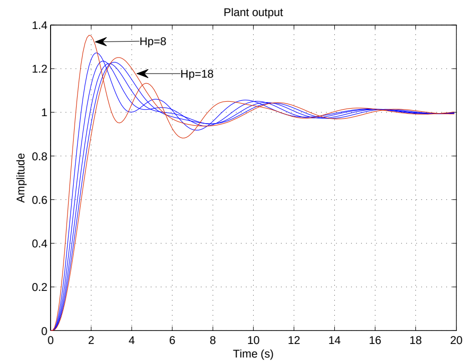

Fig. 4. The effect of the prediction horizon on the integer model MPC

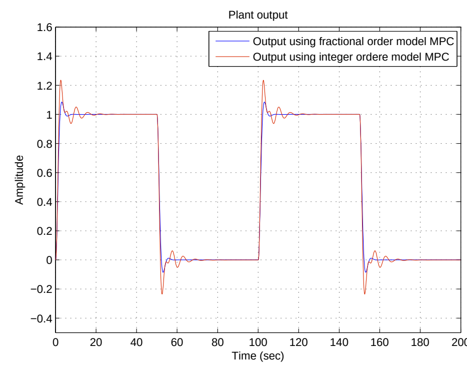

Fig. 5. Controlled plant output using both fractional model and integer order MPC

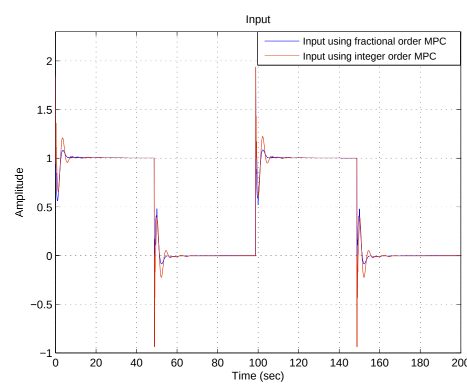

Fig. 6. Control input signal using both fractional model and integer order MPC

tional model and integer order MPC

5. CONCLUSION

In this paper, a fractional order MPC is proposed for fractional order systems control. The benefits of fractional order models for real dynamical objects and processes become more and more obvious. Through the fractional order and integer order dynamical models, the proposed fractional order MPC has been presented. The simulation results illustrate that the use of fractional order models to control systems that present fractional dynamic behaviors, to construct a fractional model MPC achieves better control performances compared to those of the conventional

MPC. This approach allows an efficient formulation of MPC while guaranteeing stability and performance of the closed-loop control system.

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