

Fractional Exact Solutions and Solitons in Gravity [★]

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Abstract: We survey our recent results on fractional gravity theory. It is also provided the Main Theorem on encoding of geometric data (metrics and connections in gravity and geometric mechanics) into solitonic hierarchies. Our approach is based on Caputo fractional derivative and nonlinear connection formalism.

Keywords: Fractional calculus, fractional geometry, fractional gravity, nonlinear connection, nonholonomic manifold.

1. INTRODUCTION

Recently, we extended the fractional calculus to Ricci flow theory, gravity and geometric mechanics, solitonic hierarchies etc (1; 2; 3; 4; 5; 6). In this work, we outline some basic geometric constructions related to fractional derivatives and integrals and their applications in modern physics and mechanics.

Our approach is also connected to a method when non-holonomic deformations of geometric structures ¹ induce a canonical connection, adapted to a necessary type nonlinear connection structure, for which the matrix coefficients of curvature are constant (7; 8). For such an auxiliary connection, it is possible to define a bi-Hamiltonian structure and derive the corresponding solitonic hierarchy.

The paper is organized as follows: In section 2, we outline the geometry of N-adapted fractional manifolds and provide an introduction to fractional gravity. In section 3, we show how fractional gravitational field equations can be solved in a general form. Section 4 is devoted to the Main Theorem on fractional solitonic hierarchies corresponding to metrics and connections in fractional gravity. The Appendix contains necessary definitions and formulas on Caputo fractional derivatives.

2. FRACTIONAL NONHOLONOMIC MANIFOLDS AND GRAVITY

Let us consider a "prime" nonholonomic manifold \mathbf{V} is of integer dimension $\dim \mathbf{V} = n + m, n \geq 2, m \geq 1$. ²

[★] ¹ determined by a fundamental Lagrange/ Finsler / Hamilton generating function (or, for instance, and Einstein metric)

² A nonholonomic manifold is a manifold endowed with a non-integrable (equivalently, nonholonomic, or anholonomic) distribution.

Its fractional extension ${}^\alpha \mathbf{V}$ is modelled by a quadruple $(\mathbf{V}, {}^\alpha \mathbf{N}, {}^\alpha \mathbf{d}, {}^\alpha \mathbf{I})$, where ${}^\alpha \mathbf{N}$ is a nonholonomic distribution stating a nonlinear connection (N-connection) structure (for details, see Appendix A with explanations for formula (1). The fractional differential structure ${}^\alpha \mathbf{d}$ is determined by Caputo fractional derivative (A.1) following formulas (A.3) and (A.4). The non-integer integral structure ${}^\alpha \mathbf{I}$ is defined by rules of type (A.2).

A nonlinear connection (N-connection) ${}^\alpha \mathbf{N}$ for a fractional space ${}^\alpha \mathbf{V}$ is defined by a nonholonomic distribution (Whitney sum) with conventional h- and v-subspaces, $\underline{h} {}^\alpha \mathbf{V}$ and $\underline{v} {}^\alpha \mathbf{V}$,

$${}^\alpha T {}^\alpha \mathbf{V} = \underline{h} {}^\alpha \mathbf{V} \oplus \underline{v} {}^\alpha \mathbf{V}. \quad (1)$$

A fractional N-connection is defined by its local coefficients ${}^\alpha \mathbf{N} = \{ {}^\alpha N_i^a \}$, when

$${}^\alpha \mathbf{N} = {}^\alpha N_i^a(u)(dx^i)^\alpha \otimes {}^\alpha \underline{\partial}_a. \quad (2)$$

For a N-connection ${}^\alpha \mathbf{N}$, we can always construct a class of fractional (co) frames (N-adapted) linearly depending on ${}^\alpha N_i^a$,

$${}^\alpha \mathbf{e}_\beta = [{}^\alpha \mathbf{e}_j = {}^\alpha \underline{\partial}_j - {}^\alpha N_j^a {}^\alpha \underline{\partial}_a, {}^\alpha \mathbf{e}_b = {}^\alpha \underline{\partial}_b], \quad (3)$$

$${}^\alpha \mathbf{e}^\beta = [{}^\alpha \mathbf{e}^j = (dx^j)^\alpha, {}^\alpha \mathbf{e}^b = (dy^b)^\alpha + {}^\alpha N_k^b (dx^k)^\alpha]. \quad (4)$$

The nontrivial nonholonomy coefficients are computed

$${}^\alpha W_{ib}^a = {}^\alpha \underline{\partial}_b {}^\alpha N_i^a, {}^\alpha W_{ij}^a = {}^\alpha \Omega_{ji}^a = {}^\alpha \mathbf{e}_i {}^\alpha N_j^a - {}^\alpha \mathbf{e}_j {}^\alpha N_i^a$$

for

$$[{}^\alpha \mathbf{e}_\alpha, {}^\alpha \mathbf{e}_\beta] = {}^\alpha \mathbf{e}_\alpha {}^\alpha \mathbf{e}_\beta - {}^\alpha \mathbf{e}_\beta {}^\alpha \mathbf{e}_\alpha = {}^\alpha W_{\alpha\beta}^\gamma {}^\alpha \mathbf{e}_\gamma. \quad (5)$$

In above formulas, the values ${}^\alpha \Omega_{ji}^a$ are called the coefficients of N-connection curvature. A nonholonomic manifold defined by a structure ${}^\alpha \mathbf{N}$ is called, in brief, a N-anholonomic fractional manifold.

We write a metric structure ${}^\alpha \mathbf{g} = \{ {}^\alpha g_{\alpha\beta} \}$ on ${}^\alpha \mathbf{V}$ in the form

$$\begin{aligned} {}^\alpha \mathbf{g} &= {}^\alpha g_{kj}(x, y) {}^\alpha e^k \otimes {}^\alpha e^j + {}^\alpha g_{cb}(x, y) {}^\alpha e^c \otimes {}^\alpha e^b \\ &= \eta_{k'j'} {}^\alpha e^{k'} \otimes {}^\alpha e^{j'} + \eta_{c'b'} {}^\alpha e^{c'} \otimes {}^\alpha e^{b'}, \end{aligned} \quad (6)$$

where matrices $\eta_{k'j'} = \text{diag}[\pm 1, \pm 1, \dots, \pm 1]$ and $\eta_{a'b'} = \text{diag}[\pm 1, \pm 1, \dots, \pm 1]$, for the signature of a "prime" space-time \mathbf{V} , are obtained by frame transforms $\eta_{k'j'} = e^k_{k'} e^j_{j'}$ and $\eta_{a'b'} = e^a_{a'} e^b_{b'}$.

A distinguished connection (d-connection) ${}^\alpha \mathbf{D}$ on ${}^\alpha \mathbf{V}$ is defined as a linear connection preserving under parallel transports the Whitney sum (1). We can associate a N-adapted differential 1-form

$${}^\alpha \mathbf{T}^\tau_\beta = {}^\alpha \mathbf{T}^\tau_{\beta\gamma} {}^\alpha e^\gamma, \quad (7)$$

parametrizing the coefficients (with respect to (4) and (3)) in the form ${}^\alpha \mathbf{T}^\tau_{\beta\gamma} = ({}^\alpha L^i_{jk}, {}^\alpha L^a_{bk}, {}^\alpha C^i_{jc}, {}^\alpha C^a_{bc})$.

The absolute fractional differential ${}^\alpha \mathbf{d} = {}_{1x} {}^\alpha d_x + {}_{1y} {}^\alpha d_y$ acts on fractional differential forms in N-adapted form; the value ${}^\alpha \mathbf{d} := {}^\alpha e^\beta {}^\alpha e_\beta$ splits into exterior h- and v-derivatives when

$$\begin{aligned} {}_{1x} {}^\alpha d_x &:= (dx^i)^\alpha {}_{1x} {}^\alpha \partial_i = {}^\alpha e^j {}^\alpha e_j \text{ and} \\ {}_{1y} {}^\alpha d_y &:= (dy^a)^\alpha {}_{1x} {}^\alpha \partial_a = {}^\alpha e^b {}^\alpha e_b. \end{aligned}$$

The torsion and curvature of a fractional d-connection ${}^\alpha \mathbf{D} = \{ {}^\alpha \mathbf{T}^\tau_{\beta\gamma} \}$ can be defined and computed, respectively, as fractional 2-forms,

$$\begin{aligned} {}^\alpha \mathcal{T}^\tau &:= {}^\alpha \mathbf{D} {}^\alpha e^\tau = {}^\alpha \mathbf{d} {}^\alpha e^\tau + {}^\alpha \mathbf{T}^\tau_{\beta\gamma} {}^\alpha e^\beta {}^\alpha e^\gamma, \\ {}^\alpha \mathcal{R}^\tau_{\beta\gamma} &:= {}^\alpha \mathbf{D} {}^\alpha \mathbf{T}^\tau_{\beta\gamma} = {}^\alpha \mathbf{d} {}^\alpha \mathbf{T}^\tau_{\beta\gamma} - {}^\alpha \mathbf{T}^\tau_{\beta\gamma} \wedge {}^\alpha \mathbf{T}^\tau_{\gamma\delta} \\ &= {}^\alpha \mathbf{R}^\tau_{\beta\gamma\delta} {}^\alpha e^\delta. \end{aligned} \quad (8)$$

There are two another important geometric objects: the fractional Ricci tensor ${}^\alpha \mathcal{R}ic = \{ {}^\alpha \mathbf{R}_{\alpha\beta} : {}^\alpha \mathbf{R}^\tau_{\alpha\beta\tau} \}$ with components

$$\begin{aligned} {}^\alpha R_{ij} &:= {}^\alpha R^k_{ijk}, \quad {}^\alpha R_{ia} := - {}^\alpha R^k_{ika}, \\ {}^\alpha R_{ai} &:= {}^\alpha R^b_{aib}, \quad {}^\alpha R_{ab} := {}^\alpha R^c_{abc} \end{aligned} \quad (9)$$

and the scalar curvature of fractional d-connection ${}^\alpha \mathbf{D}$,

$$\begin{aligned} {}^\alpha \mathbf{R} &:= {}^\alpha \mathbf{g}^{\tau\beta} {}^\alpha \mathbf{R}_{\tau\beta} = {}^\alpha R + {}^\alpha S, \\ {}^\alpha R &= {}^\alpha g^{ij} {}^\alpha R_{ij}, \quad {}^\alpha S = {}^\alpha g^{ab} {}^\alpha R_{ab}, \end{aligned} \quad (10)$$

with ${}^\alpha \mathbf{g}^{\tau\beta}$ being the inverse coefficients to a d-metric (6).

We can introduce the Einstein tensor ${}^\alpha \mathcal{E}ns = \{ {}^\alpha \mathbf{G}_{\alpha\beta} \}$,

$${}^\alpha \mathbf{G}_{\alpha\beta} := {}^\alpha \mathbf{R}_{\alpha\beta} - \frac{1}{2} {}^\alpha \mathbf{g}_{\alpha\beta} {}^\alpha \mathbf{R}. \quad (11)$$

For various applications, we can consider more special classes of d-connections:

- There is a unique canonical metric compatible fractional d-connection

$${}^\alpha \widehat{\mathbf{D}} = \{ {}^\alpha \widehat{\mathbf{T}}^\gamma_{\alpha\beta} = ({}^\alpha \widehat{L}^i_{jk}, {}^\alpha \widehat{L}^a_{bk}, {}^\alpha \widehat{C}^i_{jc}, {}^\alpha \widehat{C}^a_{bc}) \},$$

when ${}^\alpha \widehat{\mathbf{D}} ({}^\alpha \mathbf{g}) = 0$, satisfying the conditions ${}^\alpha \widehat{T}^i_{jk} = 0$ and ${}^\alpha \widehat{T}^a_{bc} = 0$, but ${}^\alpha \widehat{T}^i_{ja}, {}^\alpha \widehat{T}^a_{ji}$ and ${}^\alpha \widehat{T}^a_{bi}$

are not zero. The N-adapted coefficients are given in explicitly form in our works (1; 2; 3; 4; 5; 6).

- The fractional Levi-Civita connection ${}^\alpha \nabla = \{ {}^\alpha \Gamma^\gamma_{\alpha\beta} \}$ can be defined in standard form but for the fractional Caputo left derivatives acting on the coefficients of a fractional metric.

On spaces with nontrivial nonholonomic structure, it is preferred to work on ${}^\alpha \mathbf{V}$ with ${}^\alpha \widehat{\mathbf{D}} = \{ {}^\alpha \widehat{\mathbf{T}}^\gamma_{\tau\beta} \}$ instead of ${}^\alpha \nabla$ (the last one is not adapted to the N-connection splitting (1)). The torsion ${}^\alpha \widehat{\mathbf{T}}^\tau$ (8) of ${}^\alpha \widehat{\mathbf{D}}$ is uniquely induced nonholonomically by off-diagonal coefficients of the d-metric (6).

With respect to N-adapted fractional bases (3) and (4), the coefficients of the fractional Levi-Civita and canonical d-connection satisfy the distorting relations

$${}^\alpha \Gamma^\gamma_{\alpha\beta} = {}^\alpha \widehat{\mathbf{T}}^\gamma_{\alpha\beta} + {}^\alpha Z^\gamma_{\alpha\beta}, \quad (12)$$

where the N-adapted coefficients of distortion tensor ${}^\alpha Z^\gamma_{\alpha\beta}$ are computed in Ref. (6).

An unified approach to Einstein-Lagrange/Finsler gravity for arbitrary integer and non-integer dimensions is possible for the fractional canonical d-connection ${}^\alpha \widehat{\mathbf{D}}$. The fractional gravitational field equations are formulated for the Einstein d-tensor (11), following the same principle of constructing the matter source ${}^\alpha \mathbf{T}_{\beta\delta}$ as in general relativity but for fractional metrics and d-connections,

$${}^\alpha \widehat{\mathbf{E}}_{\beta\delta} = {}^\alpha \mathbf{T}_{\beta\delta}. \quad (13)$$

Such a system of integro-differential equations for generalized connections can be restricted to fractional nonholonomic configurations for ${}^\alpha \nabla$ if we impose the additional constraints

$${}^\alpha \widehat{L}^c_{aj} = {}^\alpha e_a ({}^\alpha N^c_j), \quad {}^\alpha \widehat{C}^i_{jb} = 0, \quad {}^\alpha \Omega^a_{ji} = 0. \quad (14)$$

There are not theoretical or experimental evidences that for fractional dimensions we must impose conditions of type (14) but they have certain physical motivation if we develop models which in integer limits result in the general relativity theory.

3. EXACT SOLUTIONS IN FRACTIONAL GRAVITY

We studied in detail (2) what type of conditions must satisfy the coefficients of a metric (6) for generating exact solutions of the fractional Einstein equations (13). For simplicity, we can use a "prime" dimension splitting of type 2 + 2 when coordinated are labeled in the form $u^\beta = (x^j, y^3 = v, y^4)$, for $i, j, \dots = 1, 2$. and the metric ansatz has one Killing symmetry when the coefficients do not depend explicitly on variable y^4 .

3.1 Separation of equations for fractional and integer dimensions

The solutions of equations can be constructed for a general source of type³

³ such parametrizations of energy-momentum tensors are quite general ones for various types of matter sources

$$\begin{aligned} {}^\alpha\Upsilon_\beta &= \text{diag}[{}^\alpha\Upsilon_\gamma; {}^\alpha\Upsilon_1 = {}^\alpha\Upsilon_2 = {}^\alpha\Upsilon_2(x^k, v); \\ &{}^\alpha\Upsilon_3 = {}^\alpha\Upsilon_4 = {}^\alpha\Upsilon_4(x^k)] \end{aligned}$$

For such sources and ansatz with Killing symmetries for metrics, the Einstein equations (13) can be integrated in general form.

We can construct 'non-Killing' solutions depending on all coordinates when

$$\begin{aligned} {}^\alpha\mathbf{g} &= {}^\alpha g_i(x^k) {}^\alpha dx^i \otimes {}^\alpha dx^i + \\ &{}^\alpha \omega^2(x^j, v, y^4) {}^\alpha h_a(x^k, v) {}^\alpha \mathbf{e}^a \otimes {}^\alpha \mathbf{e}^a, \\ {}^\alpha \mathbf{e}^3 &= {}^\alpha dy^3 + {}^\alpha w_i(x^k, v) {}^\alpha dx^i, \\ {}^\alpha \mathbf{e}^4 &= {}^\alpha dy^4 + {}^\alpha n_i(x^k, v) {}^\alpha dx^i, \end{aligned} \quad (15)$$

for any ${}^\alpha\omega$ for which

$${}^\alpha \mathbf{e}_k {}^\alpha \omega = {}^\alpha \underline{\partial}_k {}^\alpha \omega + {}^\alpha w_k {}^\alpha \omega^* + {}^\alpha n_k {}^\alpha \underline{\partial}_{y^4} {}^\alpha \omega = 0. \quad (16)$$

Configurations with fractional Levi-Civita connection ${}^\alpha\nabla$, of type (14), can be extracted by imposing additional constraints

$$\begin{aligned} {}^\alpha w_i^* &= {}^\alpha \mathbf{e}_i \ln |{}^\alpha h_4|, \quad {}^\alpha \mathbf{e}_k {}^\alpha w_i = {}^\alpha \mathbf{e}_i {}^\alpha w_k, \\ {}^\alpha n_i^* &= 0, \quad {}^\alpha \underline{\partial}_i {}^\alpha n_k = {}^\alpha \underline{\partial}_k {}^\alpha n_i, \end{aligned} \quad (17)$$

where the partial derivatives are

$$\begin{aligned} {}^\alpha a^\bullet &= {}^\alpha \underline{\partial}_1 a = {}_{1x^1} {}^\alpha \underline{\partial}_{x^1} a, \\ {}^\alpha a' &= {}^\alpha \underline{\partial}_2 a = {}_{1x^2} {}^\alpha \underline{\partial}_{x^2} a, \quad {}^\alpha a^* = {}^\alpha \underline{\partial}_v a = {}_{1v} {}^\alpha \underline{\partial}_v a, \end{aligned}$$

being used the left Caputo fractional derivatives (A.3).

3.2 Solutions with ${}^\alpha h_{3,4}^* \neq 0$ and ${}^\alpha \Upsilon_{2,4} \neq 0$

For simplicity, we provide only a class of exact solution with metrics of type (15) when ${}^\alpha h_{3,4}^* \neq 0$ (in Ref. (2), there are analyzed all possibilities for coefficients⁴) We consider the ansatz

$$\begin{aligned} {}^\alpha\mathbf{g} &= e^{{}^\alpha\psi(x^k)} {}^\alpha dx^i \otimes {}^\alpha dx^i + h_3(x^k, v) {}^\alpha \mathbf{e}^3 \otimes {}^\alpha \mathbf{e}^3 \\ &+ h_4(x^k, v) {}^\alpha \mathbf{e}^4 \otimes {}^\alpha \mathbf{e}^4, \\ {}^\alpha \mathbf{e}^3 &= {}^\alpha dv + {}^\alpha w_i(x^k, v) {}^\alpha dx^i, \\ {}^\alpha \mathbf{e}^4 &= {}^\alpha dy^4 + {}^\alpha n_i(x^k, v) {}^\alpha dx^i \end{aligned} \quad (18)$$

We consider any nonconstant ${}^\alpha\phi = {}^\alpha\phi(x^i, v)$ as a generating function. We have to solve respectively the two dimensional fractional Laplace equation, for ${}^\alpha g_1 = {}^\alpha g_2 = e^{{}^\alpha\psi(x^k)}$. Then we integrate on v , in order to determine ${}^\alpha h_3$, ${}^\alpha h_4$ and ${}^\alpha n_i$, and solve algebraic equations, for ${}^\alpha w_i$. We obtain (computing consequently for a chosen ${}^\alpha\phi(x^k, v)$)

$$\begin{aligned} {}^\alpha g_1 &= {}^\alpha g_2 = e^{{}^\alpha\psi(x^k)}, \quad {}^\alpha h_3 = \pm \frac{|{}^\alpha\phi^*(x^i, v)|}{{}^\alpha\Upsilon_2}, \quad (19) \\ {}^\alpha h_4 &= {}^\alpha_0 h_4(x^k) \pm 2 {}_{1v} {}^\alpha I_v \frac{(\exp[2 {}^\alpha\phi(x^k, v)])^*}{{}^\alpha\Upsilon_2}, \\ {}^\alpha w_i &= - {}^\alpha \underline{\partial}_i {}^\alpha\phi / {}^\alpha\phi^*, \\ {}^\alpha n_i &= {}^\alpha_1 n_k(x^i) + {}^\alpha_2 n_k(x^i) {}_{1v} {}^\alpha I_v [{}^\alpha h_3 / (\sqrt{|{}^\alpha h_4|})^3], \end{aligned}$$

⁴ by nonholonomic transforms, various classes of solutions can be transformed from one to another

where ${}^\alpha_0 h_4(x^k)$, ${}^\alpha_1 n_k(x^i)$ and ${}^\alpha_2 n_k(x^i)$ are integration functions, and ${}_{1v} {}^\alpha I_v$ is the fractional integral on variables v and

$$\begin{aligned} {}^\alpha\phi &= \ln \left| \frac{{}^\alpha h_4^*}{\sqrt{|{}^\alpha h_3 {}^\alpha h_4|}} \right|, \quad {}^\alpha\gamma = \left(\ln |{}^\alpha h_4|^{3/2} / |{}^\alpha h_3| \right)^*, \\ {}^\alpha\alpha_i &= {}^\alpha h_4^* {}^\alpha \underline{\partial}_k {}^\alpha\phi, \quad {}^\alpha\beta = {}^\alpha h_4^* {}^\alpha\phi^*. \end{aligned} \quad (20)$$

For ${}^\alpha h_4^* \neq 0$; ${}^\alpha\Upsilon_2 \neq 0$, we have ${}^\alpha\phi^* \neq 0$. The exponent $e^{{}^\alpha\psi(x^k)}$ is the fractional analog of the "integer" exponential functions and called the Mittag-Leffler function $E_\alpha[(x - {}^1x)^\alpha]$. For ${}^\alpha\psi(x) = E_\alpha[(x - {}^1x)^\alpha]$, we have ${}^\alpha \underline{\partial}_i E_\alpha = E_\alpha$.

We have to constrain the coefficients (19) to satisfy the conditions (17) in order to construct exact solutions for the Levi-Civita connection ${}^\alpha\nabla$. To select such classes of solutions, we can fix a nonholonomic distribution when ${}^\alpha_2 n_k(x^i) = 0$ and ${}^\alpha_1 n_k(x^i)$ are any functions satisfying the conditions ${}^\alpha \underline{\partial}_i {}^\alpha_1 n_k(x^j) = {}^\alpha \underline{\partial}_k {}^\alpha_1 n_i(x^j)$. The constraints on ${}^\alpha\phi(x^k, v)$ are related to the N-connection coefficients ${}^\alpha w_i = - {}^\alpha \underline{\partial}_i {}^\alpha\phi / {}^\alpha\phi^*$ following relations

$$\begin{aligned} ({}^\alpha w_i[{}^\alpha\phi])^* + {}^\alpha w_i[{}^\alpha\phi] ({}^\alpha h_4[{}^\alpha\phi])^* + {}^\alpha \underline{\partial}_i {}^\alpha h_4[{}^\alpha\phi] &= 0, \\ {}^\alpha \underline{\partial}_i {}^\alpha w_k[{}^\alpha\phi] &= {}^\alpha \underline{\partial}_k {}^\alpha w_i[{}^\alpha\phi], \end{aligned}$$

where, for instance, we denoted by ${}^\alpha h_4[{}^\alpha\phi]$ the functional dependence on ${}^\alpha\phi$. Such conditions are always satisfied for ${}^\alpha\phi = {}^\alpha\phi(v)$ or if ${}^\alpha\phi = \text{const}$ when ${}^\alpha w_i(x^k, v)$ can be any functions with zero ${}^\alpha\beta$ and ${}^\alpha\alpha_i$, see (20)).

4. THE MAIN THEOREM ON FRACTIONAL SOLITONIC HIERARCHIES

In Ref. (6; 7; 8), we proved that the geometric data for any fractional metric (in a model of fractional gravity or geometric mechanics) naturally define a N-adapted fractional bi-Hamiltonian flow hierarchy inducing anholonomic fractional solitonic configurations.

Theorem: For any N -anholonomic fractional manifold with prescribed fractional d -metric structure, there is a hierarchy of bi-Hamiltonian N -adapted fractional flows of curves $\gamma(\tau, \mathbf{l}) = h\gamma(\tau, \mathbf{l}) + v\gamma(\tau, \mathbf{l})$ described by geometric nonholonomic fractional map equations. The 0 fractional flows are defined as convective (traveling wave) maps $\gamma_\tau = \gamma_{\mathbf{l}}$, distinguished $(h\gamma)_\tau = (h\gamma)_{h\mathbf{X}}$ and $(v\gamma)_\tau = (v\gamma)_{v\mathbf{X}}$. There are fractional +1 flows defined as non-stretching mKdV maps

$$\begin{aligned} -(h\gamma)_\tau &= {}^\alpha \mathbf{D}_{h\mathbf{X}}^2 (h\gamma)_{h\mathbf{X}} + \frac{3}{2} |{}^\alpha \mathbf{D}_{h\mathbf{X}} (h\gamma)_{h\mathbf{X}}|_{hg}^2 (h\gamma)_{h\mathbf{X}}, \\ -(v\gamma)_\tau &= {}^\alpha \mathbf{D}_{v\mathbf{X}}^2 (v\gamma)_{v\mathbf{X}} + \frac{3}{2} |{}^\alpha \mathbf{D}_{v\mathbf{X}} (v\gamma)_{v\mathbf{X}}|_{vg}^2 (v\gamma)_{v\mathbf{X}}, \end{aligned}$$

and fractional +2,... flows as higher order analogs. Finally, the fractional -1 flows are defined by the kernels of recursion fractional operators inducing non-stretching fractional maps ${}^\alpha \mathbf{D}_{h\mathbf{Y}} (h\gamma)_{h\mathbf{X}} = 0$ and ${}^\alpha \mathbf{D}_{v\mathbf{Y}} (v\gamma)_{v\mathbf{X}} = 0$.

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Appendix A. FRACTIONAL CAPUTO N-ANHOLONOMIC MANIFOLDS

The fractional left, respectively, right Caputo derivatives are defined by formulas

$$\begin{aligned} {}_1x \, {}^\alpha \underline{\partial}_x f(x) &:= \frac{1}{\Gamma(s-\alpha)} \\ &\int_{{}_1x}^x (x-x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'} \right)^s f(x') dx'; \\ {}_x \, {}^\alpha \underline{\partial}_{{}_2x} f(x) &:= \frac{1}{\Gamma(s-\alpha)} \\ &\int_x^{{}_2x} (x'-x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'} \right)^s f(x') dx'. \end{aligned} \quad (\text{A.1})$$

We can introduce ${}^\alpha d := (dx^j)^\alpha {}_0 \, {}^\alpha \underline{\partial}_j$ for the fractional absolute differential, where ${}^\alpha dx^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}$ if ${}_1x^i = 0$. Such formulas allow us to elaborate the concept of fractional tangent bundle ${}^\alpha \underline{T}M$, for $\alpha \in (0, 1)$, associated to a manifold M of necessary smooth class and integer $\dim M = n$.⁵

Let us denote by $L_z({}_1x, {}_2x)$ the set of those Lebesgue measurable functions f on $[{}_1x, {}_2x]$ when $\|f\|_z = (\int_{{}_1x}^{{}_2x} |f(x)|^z dx)^{1/z} < \infty$ and $C^z[{}_1x, {}_2x]$ be the space of functions which are z times continuously differentiable on this interval. For any real-valued function $f(x)$ defined on a closed interval $[{}_1x, {}_2x]$, there is a function $F(x) = {}_1x \, {}^\alpha I_x f(x)$ defined by the fractional Riemann-Liouville integral

$${}_1x \, {}^\alpha I_x f(x) := \frac{1}{\Gamma(\alpha)} \int_{{}_1x}^x (x-x')^{\alpha-1} f(x') dx',$$

when $f(x) = {}_1x \, {}^\alpha \underline{\partial}_x F(x)$, for all $x \in [{}_1x, {}_2x]$, satisfies the conditions

$$\begin{aligned} {}_1x \, {}^\alpha \underline{\partial}_x ({}_1x \, {}^\alpha I_x f(x)) &= f(x), \quad \alpha > 0, \\ {}_1x \, {}^\alpha I_x ({}_1x \, {}^\alpha \underline{\partial}_x F(x)) &= F(x) - F({}_1x), \quad 0 < \alpha < 1. \end{aligned} \quad (\text{A.2})$$

We can consider fractional (co) frame bases on ${}^\alpha \underline{T}M$. For instance, a fractional frame basis ${}^\alpha \underline{e}_\beta = e^{\beta'}_{\beta} (u^\beta) {}^\alpha \underline{\partial}_{\beta'}$ is connected via a vierlbein transform $e^{\beta'}_{\beta} (u^\beta)$ with a fractional local coordinate basis

$${}^\alpha \underline{\partial}_{\beta'} = \left({}^\alpha \underline{\partial}_{j'} = {}_1x^{j'} \, {}^\alpha \underline{\partial}_{j'}, \, {}^\alpha \underline{\partial}_{b'} = {}_1y^{b'} \, {}^\alpha \underline{\partial}_{b'} \right), \quad (\text{A.3})$$

for $j' = 1, 2, \dots, n$ and $b' = n+1, n+2, \dots, n+n$. The fractional co-bases are related via ${}^\alpha \underline{e}^\beta = e^{\beta'}_{\beta} (u^\beta) {}^\alpha du^{\beta'}$, where

$${}^\alpha du^{\beta'} = \left((dx^{i'})^\alpha, (dy^{a'})^\alpha \right). \quad (\text{A.4})$$

The fractional absolute differential ${}^\alpha d$ is written in the form ${}^\alpha d := (dx^j)^\alpha {}_0 \, {}^\alpha \underline{\partial}_j$, where

$${}^\alpha dx^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}, \quad (\text{A.5})$$

where we consider ${}_1x^i = 0$.

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⁵ For simplicity, we may write both the integer and fractional local coordinates in the form $u^\beta = (x^j, y^a)$. We underlined the symbol T in order to emphasize that we shall associate the approach to a fractional Caputo derivative.