Euler-Lagrange equations for the Caputo fractional derivative

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Abstract: Is this note we exhibit several necessary conditions of optimality for functionals containing the Caputo fractional derivative. We consider several cases: with free or fixed boundary conditions, constrained and unconstrained problems.

Keywords: Fractional calculus, Caputo fractional derivative, Euler-Lagrange equations, isoperimetric problem.

1. INTRODUCTION

Fractional calculus plays a main role in many different areas, and has proven to be a multidisciplinary subject A. A. Kilbas and Trujillo (2006); Podlubny (1999). The subject is not recent and it is as old as the calculus itself. In a letter dated 30th September 1695, Leibniz proposed the following problem to L'Hopital: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" Since then, several mathematicians studied this question, among them Liouville, Riemann, Weyl and Letnikov. There are many fields of applications where we can use the fractional calculus, like viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos and fractals. To solve fractional differential equations, there exist several methods: Laplace and Fourier transforms, truncated Taylor series, numerical methods, etc. In Almeida and Torres (2010) a new method to find exact solutions to fractional variational problems is proposed, based on an idea introduced by Leitmann. By an appropriate coordinate transformation, we rewrite the initial problem as an equivalent simpler one; knowing the solution for the new equivalent problem, and since there exists an one-to-one correspondence between the minimizers (or maximizers) of the new problem with the ones of the original problem, we determine the desired solution . There are several definitions of fractional derivatives and fractional integrals, like Riemann–Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald–Letnikov, Hadamard, Chen, etc.

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Calculus of variations is a field of mathematics that deals with functionals. Such functionals usually envolve integrals, some unknown function and some derivative of the unknown function. The fundamental problem is to find functions that extremize the functional. Since this can be a hard task to do, one wish to study necessary and sufficient conditions of optimality. The simplest example is the following: what is the shape of the curve, joining two fixed points, that has the minimum possible length? The answer is obviously a straight line. If the points are not fixed, i.e., if we are only interested in the minimum length, the answer in an horizontal straight line (free endpoints problem). Another classical problem is the so called isoperimetric problem, and can be stated in the following way. One seeks to find a continuously differentiable function $y = y(x)$, satisfying given boundary condition $y(a) = a_0$ and $y(b) = b_0$, which minimizes or maximizes a given functional

$$I(y) = \int_a^b f(x, y(x), y'(x))\, dx,$$

for which a second given functional

$$G(y) = \int_a^b g(x, y(x), y'(x))\, dx$$

possesses a given prescribed value K. For example, amongst all closed curves with given length, which one encloses the largest area? The answer, as we know, is the circle. The classical method to solve this problem involves a Lagrange multiplier λ and consider the problem of extremizing the functional

$$\int_a^b (f - \lambda g)\, dx$$

using the respective Euler-Lagrange equation.

Fractional calculus of variations is a recent field, where classical variational problems are considered but in presence of some fractional derivative or fractional integral. In the past years a lot of attention has been put on finding necessary conditions of optimality for variational problems, in case the Lagrange function contains fractional derivatives Agrawal (2007); T. M. Atanacković and Pilipović (2008); Baleanu (2008a,b); El-Nabulsi and Torres (2008); Frederico and Torres (2007), fractional derivatives and fractional integrals Almeida and Torres (2009), and other types. Although there exist many works done for fractional variational calculus with the Riemann-Liouville fractional derivative, this type of differentiation does not seem suitable for this kind of problems, due to the fact that in order for the fractional derivative of a function f to be continuous on $[a, b]$, the condition $f(a) = f(b) = 0$ must be fulfilled. For constrained boundary problems, this is a very particular case.

2. PRELIMINARIES

We review here some basic features of Caputo's fractional derivative. For proofs and more on the subject, see A. A. Kilbas and Trujillo (2006); Podlubny (1999).

Let $f : [a, b] \to \mathbb{R}$ be a function, $\alpha > 0$, $n = [\alpha] + 1$, and Γ the Euler gamma function. Then,

(1) the left and right Riemann–Liouville fractional integrals of order α are defined by
$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, $$
and
$$ {}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, $$
respectively;

(2) the left and right Riemann–Liouville fractional derivatives of order α are defined by
$$ {}_aD_x^\alpha f(x) = \frac{d^n}{dx^n} {}_aI_x^{n-\alpha} f(x), $$
and
$$ {}_xD_b^\alpha f(x) = (-1)^n \frac{d^n}{dx^n} {}_xI_b^{n-\alpha} f(x), $$
respectively;

(3) the left and right Caputo fractional derivatives of order α are defined by
$$ {}_a^CD_x^\alpha f(x) = {}_aI_x^{n-\alpha} \frac{d^n}{dx^n} f(x), $$
and
$$ {}_x^CD_b^\alpha f(x) = (-1)^n {}_xI_b^{n-\alpha} \frac{d^n}{dx^n} f(x), $$
respectively.

Theorem 1. Assume that f is of class C^n on $[a, b]$. Then its left and right Caputo derivatives are continuous on the closed interval $[a, b]$.

One of the most important results in the fractional calculus for the variational methods is the integration by parts formula. It reads in the following way.

Theorem 2. Let $\alpha > 0$, and $f : [a, b] \to \mathbb{R}$ a C^1 function. Then holds the following
$$ \int_a^b g(x) \cdot {}_a^CD_x^\alpha f(x)dx = \int_a^b f(x) \cdot {}_xD_b^\alpha g(x)dx $$
$$ + \sum_{j=0}^{n-1} \left[{}_xD_b^{\alpha+j-n} g(x) \cdot {}_xD_b^{n-1-j} f(x) \right]_a^b $$
and
$$ \int_a^b g(x) \cdot {}_x^CD_b^\alpha f(x)dx = \int_a^b f(x) \cdot {}_aD_x^\alpha g(x)dx $$
$$ + \sum_{j=0}^{n-1} \left[(-1)^{n+j} {}_aD_x^{\alpha+j-n} g(x) \cdot {}_aD_x^{n-1-j} f(x) \right]_a^b, $$
where ${}_aD_x^k g(x) = {}_aI_x^{-k} g(x)$ and ${}_xD_b^k g(x) = {}_xI_b^{-k} g(x)$ whenever $k < 0$.

In the particular case when $0 < \alpha < 1$, we get
$$ \int_a^b g(x) \cdot {}_a^CD_x^\alpha f(x)dx $$
$$ = \int_a^b f(x) \cdot {}_xD_b^\alpha g(x)dx + \left[{}_xI_b^{1-\alpha} g(x) \cdot f(x) \right]_a^b $$
and
$$ \int_a^b g(x) \cdot {}_x^CD_b^\alpha f(x)dx $$
$$ = \int_a^b f(x) \cdot {}_aD_x^\alpha g(x)dx - \left[{}_aI_x^{1-\alpha} g(x) \cdot f(x) \right]_a^b. $$
In addition, if f is such that $f(a) = f(b) = 0$, then
$$ \int_a^b g(x) \cdot {}_a^CD_x^\alpha f(x)dx = \int_a^b f(x) \cdot {}_xD_b^\alpha g(x)dx $$
and
$$ \int_a^b g(x) \cdot {}_x^CD_b^\alpha f(x)dx = \int_a^b f(x) \cdot {}_aD_x^\alpha g(x)dx. $$

Along the work we denote by $\partial_i L$, $i = 1, \ldots, m$ ($m \in \mathbb{N}$), the partial derivative of function $L : \mathbb{R}^m \to \mathbb{R}$ with respect to its ith argument.

3. EULER-LAGRANGE EQUATIONS

To fix notation, we consider $\alpha, \beta \in (0, 1)$, and $[y](x)$ stands for
$$ [y](x) := (x, y(x), {}_a^CD_x^\alpha y(x), {}_x^CD_b^\beta y(x)). $$
The space of curves that we consider is
$$ E = \left\{ y \in C^1([a, b]) \,|\, y(a) = y_a \text{ and } y(b) = y_b \right\}. $$

For the space of variation functions, because of the constraint on the boundaries, we consider
$$ V = \left\{ h \in C^1([a, b]) \,|\, h(a) = 0 \text{ and } h(b) = 0 \right\}. $$

The fundamental problem of the fractional calculus of variations is addressed in the following way: find the functions $y \in E$ that maximizes or minimizes the functional

$$J(y) = \int_a^b L[y](x)dx. \tag{1}$$

As usual, the Lagrange function L is assumed to be of class C^1 on all its arguments. We also assume that $\partial_3 L$ has continuous right Riemann–Liouville fractional derivative of order α and $\partial_4 L$ has continuous left Riemann–Liouville fractional derivative of order β.

In Agrawal (2007) a necessary condition of optimality for such functionals is proved. We remark that, although the functional only contains Caputo fractional derivatives, the fractional Euler-Lagrange equation has Riemann-Liouville fractional derivatives.

Theorem 3. Agrawal (2007) If y a minimizer or a maximizer of J on E, then y is a solution of the fractional differential equation

$$\partial_2 L[y](x) + {}_xD_b^\alpha \partial_3 L[y](x) + {}_aD_x^\beta \partial_4 L[y](x) = 0, \tag{2}$$

for all $x \in [a, b]$.

The proof follows the same pattern as in the standard variational problems. To make the paper self-contained, we include a here short proof. Given $|\epsilon| \ll 1$, consider $h \in V$ and a variation function of y of form $y + \epsilon h$. Define the real valued function $j(\epsilon)$ by

$$j(\epsilon) = J(y + \epsilon\eta)$$

$$= \int_a^b L(x, y + \epsilon h, {}_a^C D_x^\alpha y + \epsilon {}_a^C D_x^\alpha h, {}_x^C D_b^\beta y + \epsilon {}_x^C D_b^\beta h) dx.$$

Since $\epsilon = 0$ is a minimizer or a maximizer of j, $j'(0) = 0$; i.e.,

$$\int_a^b \left[\partial_2 L\, h + \partial_3 L\, {}_a^C D_x^\alpha h + \partial_4 L\, {}_x^C D_b^\beta h\right] dx = 0.$$

Integrating by parts, and by the fundamental lemma of calculus of variations, we obtain the desired result.

We remark that, when $\alpha = 1$, and the functional is the standard one, $J(y) = \int_a^b L(x, y(x), y'(x))\, dx$, this fractional Euler-Lagrange equation reduces to the standard one:

$$\partial_2 L - \frac{d}{dx}\partial_3 L = 0.$$

Solutions of equation (2) are called extremals for J.

4. THE ISOPERIMETRIC PROBLEM

The fractional isoperimetric problem is stated in the following way: among all functions $y \in E$, which minimizes or maximizes the functional J as in (1), when restricted to the fractional integral constraint

$$I(y) = \int_a^b g[y]dx = l?$$

Again, g is assumed to be of class C^1, and $\partial_3 g$ has continuous right Riemann–Liouville fractional derivative of order α and $\partial_4 g$ has continuous left Riemann–Liouville fractional derivative of order β.

Theorem 4. Let y be a minimizer or maximizer of J on E, when restricted to the set of functions such that $I(\cdot) = l$. In addition, assume that y is not an extremal for I. Then, there exists a constant λ such that y is a solution of

$$\partial_2 F + {}_xD_b^\alpha \partial_3 F + {}_aD_x^\beta \partial_4 F = 0,$$

for all $x \in [a, b]$, where $F = L + \lambda g$.

Sketch of proof Given $h_1, h_2 \in V$, $|\epsilon_1| \ll 1$ and $|\epsilon_2| \ll 1$, consider

$$j(\epsilon_1, \epsilon_2) = \int_a^b L[y + \epsilon_1 h_1 + \epsilon_2 h_2](x)dx$$

and

$$i(\epsilon_1, \epsilon_2) = \int_a^b g[y + \epsilon_1 h_1 + \epsilon_2 h_2](x)dx - l.$$

Since y is not an extremal for I, there exists a function h_2 such that

$$\left.\frac{\partial i}{\partial \epsilon_2}\right|_{(0,0)} \neq 0,$$

and by the Implicit Function Theorem, there exists a C^1 function $\epsilon_2(\cdot)$, defined in some neighborhood of zero, such that

$$i(\epsilon_1, \epsilon_2(\epsilon_1)) = 0.$$

Applying the Lagrange Multiplier Rule, there exists a constant λ for which the following holds:

$$\nabla(j(0, 0) + \lambda i(0, 0)) = \mathbf{0}.$$

Differentiating j and i at $(0, 0)$, and integrating by parts, we prove the Theorem.

For example, let $\overline{y}(x) = E_\alpha(x^\alpha)$, $x \in [0, 1]$, where E_α is the Mittag–Leffler function. Then ${}_0^C D_x^\alpha \overline{y} = \overline{y}$.

Consider the following fractional variational problem:

$$\begin{cases} J(y) = \displaystyle\int_0^1 ({}_0^C D_x^\alpha y)^2\, dx \quad \rightarrow \quad \text{extr}, \\ I(y) = \displaystyle\int_0^1 \overline{y}\, {}_0^C D_x^\alpha y\, dx = \int_0^1 (\overline{y})^2\, dx, \\ y(0) = 1 \quad \text{and} \quad y(1) = E_\alpha(1). \end{cases} \tag{3}$$

In this case, we consider

$$F(x, y, {}_0^C D_x^\alpha y, {}_x^C D_1^\beta y, \lambda) = ({}_0^C D_x^\alpha y)^2 + \lambda \overline{y}\, {}_0^C D_x^\alpha y$$

and the fractional Euler–Lagrange equation is

$$\partial_2 F + {}_xD_1^\alpha \partial_3 F + {}_xD_1^\beta \partial_4 F = 0$$

i.e.,

$${}_xD_1^\alpha (2\, {}_0^C D_x^\alpha y + \lambda \overline{y}) = 0.$$

A solution of this problem is $\lambda = -2$ and $y = \overline{y}$.

We now include the case when y may be an extremal of I.

Theorem 5. If y is a minimizer or a maximizer of J, subject to the condition $I(\cdot) = l$, then there exist two constants λ_0 and λ, not both zero, such that

$$\partial_2 K + {}_xD_b^\alpha \partial_3 K + {}_aD_x^\beta \partial_4 K = 0$$

where $K = \lambda_0 L + \lambda g$.

5. TRANSVERSALITY CONDITIONS

In the present section we study transversality conditions, where the right boundary of integration b is free , as well $y(b)$.

Let

$$F=\left\{(y,x)\in C^1([a,b])\times[a,b]\,|\,y(a)=y_a\right\}.$$

The type of functional we consider now is

$$J(y,T)=\int_a^T L[y](x)\,dx,$$

where

$$[y](x):=(x,y(x),{}_a^C D_x^\alpha y(x)).$$

Theorem 6. Suppose that (y,T) minimizes or maximizes J. Then

$$\partial_2 L[y](x)+{}_x D_T^\alpha(\partial_3 L[y](x))=0$$

on $[a,T]$, and satisfies the transversality conditions

$$\begin{cases} L[y](T)-y'(T)\left[{}_x I_T^{1-\alpha}\partial_3 L[y](x)\right]_{x=T}=0 \\ \left[{}_x I_T^{1-\alpha}\partial_3 L[y](x)\right]_{x=T}=0. \end{cases}$$

The proof follows by considering variations on the function, $y(x)+\epsilon h(x)$ and variations on time $T+\epsilon\triangle T$ as well.

Now we shall rewrite the transversality conditions in terms of the increment on time $\triangle T$ and on the consequent increment on $y(\cdot)$, $\triangle y_T$. We fix $\epsilon=1$ and variation functions h satisfying the additional condition $h'(T)=0$. Define the increment by

$$\triangle y_T=(y+h)(T+\triangle T)-y(T).$$

Doing Taylor's expansion up to first order, we obtain

$$\begin{aligned}&\triangle T\left[L[y](T)-y'(T)\left[{}_x I_T^{1-\alpha}\partial_3 L[y](x)\right]_{x=T}\right]\\ &+\triangle y_T\left[{}_x I_T^{1-\alpha}\partial_3 L[y](x)\right]_{x=T}+O(\triangle T)^2=0.\end{aligned}$$

T is fixed

Thus $\triangle T=0$ and we obtain (see Agrawal (2006), equation (25))

$$\left[{}_x I_T^{1-\alpha}\partial_3 L[y](x)\right]_{x=T}=0.$$

$y(T)$ is fixed

Thus $\triangle x_T=0$ but $\triangle T$ is arbitrary. Hence,

$$L[y](T)-y'(T)\left[{}_x I_T^{1-\alpha}\partial_3 L[y](x)\right]_{x=T}=0.$$

Terminal curve

In this case the terminal point is described by a given curve ψ, in the sense that $y(T)=\psi(T)$. Since

$$\begin{aligned}\triangle y_T &=\psi(T+\triangle T)-\psi(T)\\ &=\psi'(T)\triangle T+O(\triangle T)^2,\end{aligned}$$

we get (see Agrawal (2007), equation (29))

$$(\psi'(T)-y'(T))\left[\left[{}_x I_T^{1-\alpha}\partial_3 L[y](x)\right]_{x=T}\right]+L[y](T)=0.$$

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