

Fractional variational calculus for nondifferentiable functions: generalized natural boundary conditions

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Abstract: Fractional (non-integer) derivatives and integrals play an important role in theory and applications. The fractional calculus of variations is a rather recent subject with the first results from 1996. This paper presents necessary and sufficient optimality conditions for fractional problems of the calculus of variations with a Lagrangian density depending on the free end-points. The fractional operators are defined in the sense of Jumarie.

Keywords: calculus of variations, fractional calculus, Jumarie fractional derivatives, fractional Euler-Lagrange equations, fractional natural boundary conditions.

1. INTRODUCTION

The Fractional Calculus (FC) is one of the most interdisciplinary fields of mathematics, with many applications in physics and engineering. The history of FC goes back more than three centuries, when in 1695 the derivative of order $\alpha = 1/2$ was described by Leibniz. Since then, many different forms of fractional operators were introduced: the Grunwald-Letnikov, Riemann-Liouville, Riesz, and Caputo fractional derivatives (see, e.g., Kilbas et al. (2006); Podlubny (1999); Samko et al. (1993)), and the more recent notions of Klimek (2002), Cresson (2007), and Jumarie (2005). FC is nowadays the realm of physicists and mathematicians, who investigate the usefulness of such non-integer order derivatives and integrals in different areas of physics and mathematics (see, e.g., Carpinter and Mainardi (1997); Hilfer (2000); Kilbas et al. (2006)). It is a successful tool for describing complex quantum field dynamical systems, dissipation, and long-range phenomena that cannot be well illustrated using ordinary differential and integral operators (see, e.g., El-Nabulsi and Torres (2008); Hilfer (2000); Klimek (2002); Riewe (1996)). Applications of FC are found, e.g., in classical and quantum mechanics, field theories, variational calculus, and optimal control (see, e.g., El-Nabulsi and Torres (2007); Frederico and Torres (2008); Jumarie (2007)).

The calculus of variations is an old branch of optimization theory that has many applications both in physics and geometry. Apart from a few examples known since ancient times such as Queen Dido's problem (reported in *The Aeneid* by Virgil), the problem of finding optimal curves and surfaces has been posed first by physicists such as Newton, Huygens, and Galileo. Their contemporary mathematicians, starting with the Bernoulli brothers and Leibniz, followed by Euler and Lagrange, invented the calculus of variations of a functional in order to solve those problems. Fractional Calculus of Variations (FCV) unifies

the calculus of variations and the fractional calculus, by inserting fractional derivatives into the variational integrals. This occurs naturally in many problems of physics or mechanics, in order to provide more accurate models of physical phenomena. The FCV started in 1996 with the work of Riewe (1996). Riewe formulated the problem of the calculus of variations with fractional derivatives and obtained the respective Euler-Lagrange equations, combining both conservative and nonconservative cases. Nowadays the FCV is a subject under strong research. Different definitions for fractional derivatives and integrals are used, depending on the purpose under study. Investigations cover problems depending on Riemann-Liouville fractional derivatives (see, e.g., Atanacković et al. (2008); El-Nabulsi and Torres (2008); Frederico and Torres (2007)), the Caputo fractional derivative (see, e.g., Agrawal (2006); Baleanu (2008a,b); Malinowska and Torres (2010a)), the symmetric fractional derivative (see, e.g., Klimek (2002)), the Jumarie fractional derivative (see, e.g., Almeida and Torres (2010); Almeida et al. (2010); Jumarie (2005, 2006, 2007, 2009, 2010)), and others Almeida and Torres (2009); Cresson (2007); El-Nabulsi and Torres (2007). For applications of the fractional calculus of variations we refer the reader to Dreisigmeyer and Young (2003, 2004); El-Nabulsi and Torres (2008); Jumarie (2007); Klimek (2002); Rabei and Ababneh (2008); Rabei et al. (2007); Stanislavsky (2006). Although the literature of FCV is already vast, much remains to be done.

In this paper we study problems of FCV which are defined in terms of the Jumarie fractional derivatives and integrals. The Euler-Lagrange equations for such problems with and without constraints was recently shown in Almeida and Torres (2010). Here we develop further the theory by proving necessary optimality conditions for more general problems of FCV with a Lagrangian that may also depend on the unspecified end-points $y(a)$, $y(b)$. More precisely, the problem under our study: to extremize a functional

which is defined in terms of the Jumarie fractional operators and having no constraint on $y(a)$ and/or $y(b)$. The novelty is the dependence of the integrand L on the a priori unknown final values $y(a)$, $y(b)$. The new natural boundary conditions (5)–(6) have important implications in economics (see Cruz et al. (2010) and the references therein).

The paper is organized as follows. Section 2 presents the necessary definitions and concepts of Jumarie's fractional calculus. Our results are formulated, proved, and illustrated through examples in Section 3. Main results of the paper include necessary optimality conditions with the generalized natural boundary conditions (Theorem 2) that become sufficient under appropriate convexity assumptions (Theorem 6). We finish with Section 4 of conclusions.

2. FRACTIONAL CALCULUS

For an introduction to the classical fractional calculus we refer the reader to Kilbas et al. (2006); Miller and Ross (1993); Podlubny (1999); Samko et al. (1993). In this section we briefly review the main notions and results from the recent fractional calculus proposed by Jumarie (2005, 2006, 2007, 2009). Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function and $\alpha \in (0, 1)$. The Jumarie fractional derivative of f is defined by

$$f^{(\alpha)}(x) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} (f(t) - f(0)) dt. \quad (1)$$

Note that the Jumarie and the Riemann-Liouville fractional derivatives are equal if $f(0) = 0$. The advantage of definition (1) with respect to the classical definition of Riemann-Liouville is that the fractional derivative of a constant is now zero, as desired.

We define the $(dt)^\alpha$ integral of f as follows:

$$\int_0^x f(t)(dt)^\alpha := \alpha \int_0^x (x-t)^{\alpha-1} f(t) dt.$$

For a motivation of this definition we refer to Jumarie (2005). One can easily generalize the previous definitions for continuous functions with a domain $[a, b]$:

$$f^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} (f(t) - f(a)) dt$$

and

$$\int_a^x f(t)(dt)^\alpha = \alpha \int_a^x (x-t)^{\alpha-1} f(t) dt.$$

In the discussion to follow, we will need the following formula of integration by parts (cf. Jumarie (2009)):

$$\begin{aligned} \int_a^b u^{(\alpha)}(x)v(x) (dx)^\alpha &= \alpha! [u(x)v(x)]_a^b \\ &\quad - \int_a^b u(x)v^{(\alpha)}(x) (dx)^\alpha, \end{aligned} \quad (2)$$

where $\alpha! := \Gamma(1+\alpha)$.

3. MAIN RESULTS

Let us consider the functional defined by

$$\mathcal{J}(y) = \int_a^b L(x, y(x), y^{(\alpha)}(x), y(a), y(b)) (dx)^\alpha,$$

where $L(\cdot, \cdot, \cdot, \cdot, \cdot) \in C^1([a, b] \times \mathbb{R}^4; \mathbb{R})$. The fractional problem of the calculus of variations under consideration has the form

$$\begin{aligned} \mathcal{J}(y) &\longrightarrow \text{extr} \\ (y(a) = y_a), \quad (y(b) = y_b) \\ y(\cdot) &\in C^0. \end{aligned} \quad (3)$$

Using parentheses around the end-point conditions means that the conditions may or may not be present.

Along the work we denote by $\partial_i L$, $i = 1, \dots, 5$, the partial derivative of function $L(\cdot, \cdot, \cdot, \cdot, \cdot)$ with respect to its i th argument.

The following lemma will be needed in the next subsection.

Lemma 1. Let g be a continuous function and assume that

$$\int_a^b g(x)h(x) (dx)^\alpha = 0$$

for every continuous function h satisfying $h(a) = h(b) = 0$. Then $g \equiv 0$.

Proof. Can be done in a similar way as the proof of the standard fundamental lemma of the calculus of variations (see, e.g., Brunt (2004)).

3.1 Necessary Conditions

Next theorem gives necessary optimality conditions for the problem (3).

Theorem 2. Let y be an extremizer to problem (3). Then, y satisfies the fractional Euler-Lagrange equation

$$\begin{aligned} \partial_2 L(x, y(x), y^{(\alpha)}(x), y(a), y(b)) \\ = \frac{d^\alpha}{dx^\alpha} \partial_3 L(x, y(x), y^{(\alpha)}(x), y(a), y(b)) \end{aligned} \quad (4)$$

for all $x \in [a, b]$. Moreover, if $y(a)$ is not specified, then

$$\begin{aligned} \int_a^b \partial_4 L(x, y(x), y^{(\alpha)}(x), y(a), y(b)) (dx)^\alpha \\ = \alpha! \partial_3 L(a, y(a), y^{(\alpha)}(a), y(a), y(b)) \end{aligned} \quad (5)$$

if $y(b)$ is not specified, then

$$\begin{aligned} \int_a^b \partial_5 L(x, y(x), y^{(\alpha)}(x), y(a), y(b)) (dx)^\alpha \\ = -\alpha! \partial_3 L(b, y(b), y^{(\alpha)}(b), y(a), y(b)). \end{aligned} \quad (6)$$

Proof. Suppose that y is an extremizer of \mathcal{J} and consider the value of \mathcal{J} at a nearby function $\tilde{y} = y + \varepsilon h$, where $\varepsilon \in \mathbb{R}$ is a small parameter and h is an arbitrary continuous function. We do not require $h(a) = 0$ or $h(b) = 0$ in case $y(a)$ or $y(b)$, respectively, is free (it is possible that both are free). Let $j(\varepsilon) = \mathcal{J}(y + \varepsilon h)$. Then a necessary condition for y to be an extremizer is given by $j'(0) = 0$. Hence,

$$\begin{aligned} \int_a^b \left[\partial_2 L(\cdot)h(x) + \partial_3 L(\cdot)h^{(\alpha)}(x) \right. \\ \left. + \partial_4 L(\cdot)h(a) + \partial_5 L(\cdot)h(b) \right] (dx)^\alpha = 0, \end{aligned} \quad (7)$$

where $(\cdot) = (x, y(x), y^{(\alpha)}(x), y(a), y(b))$. Using integration by parts (2) to the second term we get

$$\begin{aligned} & \int_a^b \left[\partial_2 L(\cdot) - \frac{d^\alpha}{dx^\alpha} \partial_3 L(\cdot) \right] (dx)^\alpha \\ & + \alpha! \partial_3 L(\cdot)|_{x=b} h(b) - \alpha! \partial_3 L(\cdot)|_{x=a} h(a) \\ & + \int_a^b \left[\partial_4 L(\cdot) h(a) + \partial_5 L(\cdot) h(b) \right] (dx)^\alpha = 0. \end{aligned} \quad (8)$$

We first consider functions h such that $h(a) = h(b) = 0$. Then, by the Lemma 1 we deduce that

$$\partial_2 L(\cdot) = \frac{d^\alpha}{dx^\alpha} \partial_3 L(\cdot)$$

for all $x \in [a, b]$. Therefore, in order for y to be an extremizer to the problem (3), y must be a solution of the fractional Euler-Lagrange equation. But if y is a solution of (4), the first integral in expression (8) vanishes, and then the condition (7) takes the form

$$\begin{aligned} & h(b) \left[\int_a^b \partial_5 L(\cdot) (dx)^\alpha + \alpha! \partial_3 L(\cdot)|_{x=b} \right] \\ & + h(a) \left[\int_a^b \partial_4 L(\cdot) dx - \alpha! \partial_3 L(\cdot)|_{x=a} \right] = 0. \end{aligned}$$

If $y(a) = y_a$ and $y(b) = y_b$ are given in the formulation of problem (3), then the latter equation is trivially satisfied since $h(a) = h(b) = 0$. When $y(b)$ is free, then equation (6) holds, when $y(a)$ is free, then (5) holds, since $h(a)$ or $h(b)$ is, respectively, arbitrary.

In the case L does not depend on $y(a)$ and $y(b)$, by Theorem 2 we obtain the following result.

Corollary 3. (Almeida and Torres, 2010, Theorem 1) Let y be an extremizer to problem

$$\mathcal{J}(y) = \int_a^b L(x, y(x), y^{(\alpha)}(x)) (dx)^\alpha \longrightarrow \text{extr.}$$

Then, y satisfies the fractional Euler-Lagrange equation

$$\partial_2 L(x, y(x), y^{(\alpha)}(x)) = \frac{d^\alpha}{dx^\alpha} \partial_3 L(x, y(x), y^{(\alpha)}(x))$$

for all $x \in [a, b]$. Moreover, if $y(a)$ is not specified, then

$$\partial_3 L(a, y(a), y^{(\alpha)}(a)) = 0$$

if $y(b)$ is not specified, then

$$\partial_3 L(b, y(b), y^{(\alpha)}(b)) = 0$$

Observe that if α goes to 1, then the operators $\frac{d^\alpha}{dx^\alpha}$, $(dx)^\alpha$ could be replaced with $\frac{d}{dx}$ and dx . Thus, in this case we obtain the corresponding result in the classical context of the calculus of variations (see (Malinowska and Torres, 2010b, Corollary 1), (Cruz et al., 2010, Theorem 2.1)).

Corollary 4. (Malinowska and Torres, 2010b, Corollary 1) If y is a local extremizer for

$$\begin{aligned} \mathcal{J}(y) &= \int_a^b L(x, y(x), y'(x), y(a), y(b)) dx \longrightarrow \text{extr} \\ & (y(a) = y_a), \quad (y(b) = y_b), \end{aligned}$$

then

$$\begin{aligned} & \frac{d}{dx} \partial_3 L(x, y(x), y'(x), y(a), y(b)) = \\ & \partial_2 L(x, y(x), y'(x), y(a), y(b)) \end{aligned}$$

for all $x \in [a, b]$. Moreover, if $y(a)$ is free, then

$$\begin{aligned} & \partial_3 L(a, y(a), y'(a), y(a), y(b)) \\ & = \int_a^b \partial_5 L(x, y(x), y'(x), y(a), y(b)) dx; \end{aligned}$$

and if $y(b)$ is free, then

$$\begin{aligned} & \partial_3 L(b, y(b), y'(b), y(a), y(b)) \\ & = - \int_a^b \partial_6 L(x, y(x), y'(x), y(a), y(b)) dx. \end{aligned}$$

3.2 Sufficient Conditions

In this section we prove sufficient conditions that ensure the existence of minimum (maximum). Similarly to what happens in the classical calculus of variations, some conditions of convexity (concavity) are in order.

Definition 5. Given a function L , we say that $L(\underline{x}, y, z, t, u)$ is jointly convex (concave) in (y, z, t, u) , if $\partial_i L$, $i = 2, \dots, 5$, exist and are continuous and verify the following condition:

$$\begin{aligned} & L(x, y + y_1, z + z_1, t + t_1, u + u_1) - L(x, y, z, t, u) \\ & \geq (\leq) \partial_2 L(\cdot) y_1 + \partial_3 L(\cdot) z_1 + \partial_4 L(\cdot) t_1 + \partial_5 L(\cdot) u_1, \end{aligned}$$

where $(\cdot) = (x, y, z, t, u)$, for all (x, y, z, t, u) , $(x, y + y_1, z + z_1, t + t_1, u + u_1) \in [a, b] \times \mathbb{R}^4$.

Theorem 6. Let $L(\underline{x}, y, z, t, u)$ be a jointly convex (concave) in (y, z, t, u) . If y_0 satisfies conditions (4)–(6), then y_0 is a global minimizer (maximizer) to problem (3).

Proof. We shall give the proof for the convex case. Since L is jointly convex in (y, z, t, u) for any continuous function $y_0 + h$, we have

$$\begin{aligned} & \mathcal{J}(y_0 + h) - \mathcal{J}(y_0) \\ &= \int_a^b \left[L(x, y_0(x) + h(x), y_0^{(\alpha)}(x) + h^{(\alpha)}(x), y_0(a) + h(a), \right. \\ & \quad \left. y_0(b) + h(b)) - L(x, y_0^{(\alpha)}(x), y_0(a), y_0(b)) \right] (dx)^\alpha \\ & \geq \int_a^b \left[\partial_2 L(\cdot) h(x) + \partial_3 L(\cdot) h^{(\alpha)}(x) + \partial_4 L(\cdot) h(a) \right. \\ & \quad \left. + \partial_5 L(\cdot) h(b) \right] (dx)^\alpha \end{aligned}$$

where $(\cdot) = (x, y_0(x), y_0^{(\alpha)}(x), y_0(a), y_0(b))$. We can now proceed analogously to the proof of Theorem 2. As the result we get

$$\begin{aligned} & \mathcal{J}(y_0 + h) - \mathcal{J}(y_0) \\ & \geq \int_a^b \left[\partial_2 L(\cdot) - \frac{d^\alpha}{dx^\alpha} \partial_3 L(\cdot) \right] (dx)^\alpha \\ & h(b) \left[\int_a^b \partial_5 L(\cdot) (dx)^\alpha + \alpha! \partial_3 L(\cdot)|_{x=b} \right] \\ & + h(a) \left[\int_a^b \partial_4 L(\cdot) dx - \alpha! \partial_3 L(\cdot)|_{x=a} \right] = 0, \end{aligned}$$

since y_0 satisfies conditions (4)–(6). Therefore, we obtain $\mathcal{J}(y_0 + h) \geq \mathcal{J}(y_0)$.

3.3 Examples

We shall provide examples in order to illustrate our main results.

Example 7. Consider the following problem

$$\mathcal{J}(y) = \int_0^1 \left\{ \left[\frac{x^\alpha}{\Gamma(\alpha+1)} (y^{(\alpha)})^2 - 2x^\alpha y^{(\alpha)} \right]^2 + (y(0) - 1)^2 + (y(1) - 2)^2 \right\} (dx)^\alpha \longrightarrow \text{extr.}$$

The Euler-Lagrange equation associated to this problem is

$$-\frac{d^\alpha}{dx^\alpha} \left(2 \left[\frac{x^\alpha}{\Gamma(\alpha+1)} (y^{(\alpha)})^2 - 2x^\alpha y^{(\alpha)} \right] \cdot \left[\frac{2x^\alpha}{\Gamma(\alpha+1)} y^{(\alpha)} - 2x^\alpha \right] \right) = 0. \quad (9)$$

Let $y = x^\alpha + b$, where $b \in \mathbb{R}$. Since $y^{(\alpha)} = \Gamma(\alpha+1)$, it follows that y is a solution of (9). In order to determine b we use the generalized natural boundary conditions (5)–(6), which can be written for this problem as

$$\begin{aligned} \int_0^1 (y(0) - 1)(dx)^\alpha &= 0, \\ \int_0^1 (y(1) - 2)(dx)^\alpha &= 0. \end{aligned}$$

Hence, $\tilde{y} = x^\alpha + 1$ is a candidate solution. We remark that the \tilde{y} is not differentiable in $[0, 1]$.

Example 8. Consider the following problem

$$\mathcal{J}(y) = \int_0^1 \left[(y^{(\alpha)}(x))^2 + \gamma y^2(0) + \lambda (y(1) - 1)^2 \right] (dx)^\alpha \longrightarrow \min, \quad (10)$$

where $\gamma, \lambda \in \mathbb{R}^+$. For this problem, the fractional Euler-Lagrange equation and the generalized natural boundary conditions (see Theorem 2) are given, respectively, as

$$2 \frac{d^\alpha}{dx^\alpha} y^{(\alpha)}(x) = 0, \quad (11)$$

$$\int_0^1 \gamma y(0)(dx)^\alpha = \alpha! y^{(\alpha)}(0), \quad (12)$$

$$\int_0^1 \lambda (y(1) - 1)(dx)^\alpha = -\alpha! y^{(\alpha)}(1). \quad (13)$$

Solving equations (11)–(13) we obtain that

$$\bar{y}(x) = \frac{\gamma \lambda \alpha!}{\gamma \lambda + (\alpha!)^2 (\lambda + \gamma)} x^\alpha + \frac{(\alpha!)^2 \lambda}{\gamma \lambda + (\alpha!)^2 (\lambda + \gamma)}$$

is a candidate for minimizer. Observe that problem (10) satisfies assumptions of Theorem 6. Therefore \bar{y} is a global minimizer to this problem. We note that when α goes to 1 problem (10) tends to

$$\mathcal{K}(y) = \int_0^1 \left[(y'(x))^2 + \gamma y^2(0) + \lambda (y(1) - 1)^2 \right] dx \longrightarrow \min. \quad (14)$$

with the solution

$$y(x) = \frac{\gamma \lambda}{\gamma \lambda + \lambda + \gamma} x + \frac{\lambda}{\gamma \lambda + \lambda + \gamma}.$$

4. CONCLUSION

In recent years fractional calculus has played an important role in various fields such as mechanics, electricity, chemistry, biology, economics, modeling, identification, control theory and signal processing (see, e.g., Almeida and Torres (2009b); Baleanu et al. (2010b); Debnath

(2003); Kolwankar and Gangal (1997); Machado et al. (2010); Ortigueira and Machado (1993); Ross (1975)). The fractional operators are non-local, therefore they are suitable for constructing models possessing memory. This gives several possible applications of the FCV, e.g., in describing non-local properties of physical systems in mechanics (see, e.g., Baleanu et al. (2010); Carpinter and Mainardi (1997); Klimek (2002); Rabei and Ababneh (2008); Tarasov (2006)) or electrodynamics (see, e.g., Baleanu et al. (2009); Tarasov (2008)). The Jumarie fractional derivative is quite suitable to describe dynamics evolving in a space which exhibit coarse-grained phenomenon. When the point in this space is not infinitely thin but rather a thickness, then it would be better to replace dx by $(dx)^\alpha$, $0 < \alpha < 1$, where α characterizes the grade of the phenomenon. The fractal feature of the space is transported on time, and so both space and time are fractal. Thus, the increment of time of the dynamics of the system is not dx but $(dx)^\alpha$. In this note we generalize some previous results of the FCV (which are defined in terms of the Jumarie fractional derivatives and integrals) by proving optimality conditions for problems of FCV with a Lagrangian density depending on the free end-points. The advantage of using the Jumarie fractional derivative lies in the fact that this derivative is defined for continuous functions, not necessarily differentiable (see, Example 7). Note that the integrand in problem (5) depends upon the a priori unknown final values $y(a)$ and $y(b)$. This class of problem is not contained within the classical class of FCV and the present paper indicates how such problems may be solved.

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