Fractional Optimal Control of a Hollow Cylindrical Structure

Md. Mehedi Hasan*. X.W. Tangpong*
O.P. Agrawal**

**Department of Mechanical Engineering, Dept 2490, PO Box 6050,*
North Dakota State University, Fargo, ND 58108, USA
(Tel:701-231-8839;e-mail: annie.tangpong@ndsu.edu)
*** Department of Mechanical Engineering and Energy Processes*
Southern Illinois University Carbondale, Carbondale, IL 62901, USA
(email: om@engr.siu.edu)

Abstract: This paper presents a general formulation and numerical scheme for Fractional Optimal Control Problem (FOCP) of a distributed system in cylindrical coordinate and uses a hollow cylinder with axial symmetry as the example to demonstrate the method. The fractional derivatives are expressed in the Caputo-Sense. The performance index of FOCP is considered as a function of both the state and the control variables and the dynamic constraints are expressed by a partial fractional differential equation. The method of separation of variables is employed to find the solution of the problem and the eigenfunction approach is used to decouple the equations. The Fractional Optimal Control (FOC) equations are reduced to the Volterra-type integral equations. Only a few eigenfunctions in both radial and axial directions are sufficient for the results to converge. The time domain is discretized into several subintervals and the result is more stable for smaller time steps. Various orders of fractional derivatives are analyzed and the numerical results converge toward the analytical solutions as the order of derivative goes toward the integer value of 1.

Keywords**:** fractional calculus; fractional optimal control; cylindrical coordinate; Caputo fractional derivatives; hollow cylinder.

1. INTRODUCTION

Researchers in the last decade demonstrated that many processes in physics and engineering areas were governed more accurately by fractional order differential equations instead of traditional integer order differential equation (Oustaloup *et al.*, 2000). For example, materials with memory and hereditary effects and dynamic process, such as gas diffusion and heat conduction, modelled more accurately by fractional order models than integer order models (Zamani *et al.*, 2007; Tricaud & Chen, 2010a,b). Some other applications are in behaviors of viscoelastic materials (Bagley and Calico, 1991; Koeller, 1984, 1986; Skaar *et al.*, 1988), biomechanics (Magin, 2006), control (Oustaloup *et al.*, 2000; Xue and Chen, 2002; Manabe, 2003, Monje *et al.*, 2004), electrochemical processes (Ichise *et al.*, 1971; Sun *et al.*, 1984a), dielectric polarization (Sun *et al.*, 1984b), colored noise (Mandelbrot, 1967) and chaos (Hartley *et al.*, 1995), etc. Miller and Rose (1993) mentioned that almost every field of science and engineering has the application of fractional derivatives and some of the applications are documented in (Agrawal, 2008a; Hartley & Lorenzo, 2002; Tricaud & Chen, 2010b). Machado *et al.* (2010) also presented several applications of fractional-order derivatives in science and engineering. With the emerging number of its application, fractional order calculus and its application has become an important topic for researchers in various engineering fields.

The general definition of an optimal control problem requires the minimization of a functional over an admissible set of control functions subject to dynamic constraints on the state and control variables (Agrawal, 1989). Optimal control problems have found applications in many areas including engineering, science and economics. A fractional optimal control problem (FOCP) is an optimal control problem in which either the performance index or the differential equations governing the dynamics of the system or both contain at least one fractional order derivative term (Tangpong & Agrawal, 2009). As the demand for accurate and high precision systems increases, the demand for numerical formulation and solution scheme of fraction optimal control theories also increases.

The formulation of Fractional Optimal Control Problems (FOCPs) stems from the fractional variational calculus (FVC) and this FVC is applied to deterministic and stochastic analysis of FOCPs (Baleanu *et al.,* 2009). Riewe (1996 & 1997) was among the earliest researchers to formulate a FVC, and used fractional calculus of variations to develop Larangian, Euler-Lagrange equations, and other concept for mechanics of nonconservative systems. Agrawal (2002) also presented fractional Euler-Lagrange equations for Fractional

Variational Problems (FVPs) and extended the variational calculus to fractional variational problems.

Integer order optimal controls (IOOCs) have already been well established and a significant amount of work has been done in the field of optimal control of integer order systems. Excellent textbooks are available in that field (Bryson & Ho, 1969; Sage & White, 1977; Hentenes, 1966) and various methods have been employed to solve such problems (Agrawal, 1989; Gregory & Lin, 1992). A lot of work has been done in the area of Fractional Order Control (FORC) (Manabe, 2003; Bode, 1945; Oustaloup, 1983 & 1991; Podlubny, 1999; Vinagre & Chen, 2002) without any discussions about FOCP. With the growing number of applications of FOCPs, it is necessary to establish solutions of FOCPs. Agrawal (2004) was among the earliest to establish formulations and solution schemes for FOCPs.

The area of FOCPs has grown rapidly over the last decade. Agrawal (2006) gave a general formulation of FOCPs in the Riemann-Liouville (RL) sense and described a solution scheme for FOCPs for classical optimal control problem that was based on variational virtual work coupled with the Lagrange multiplier technique. The works presented in (Tangpong & Agrawal, 2009; Agrawal, 2006; Agrawal, 2008a) formulated FOCPs in terms of Caputo fractional derivatives (CFDs) instead of RL derivatives and an iterative numerical scheme was applied to solve the problem numerically where the time domain was discretized into small segments. CFDs allow one to incorporate the usual initials conditions in a simple manner, and therefore are popular choices for researchers. In (Agrawal & Baleanu, 2007), the fractional derivatives of the system were approximated using the Grunwald-Letnikov definition that led to a set of algebraic equations that can be solved using numerical techniques. Agrawal (Agrawal, 2005) presented a general scheme for stochastic analysis of FOCPs. In (Baleanu *et al.*, 2009) a different solution scheme was proposed where a modified Grunwald-Letnikov definition was used to derive a central difference formula. Based on the expansion formula for fractional derivatives, a new solution scheme was proposed in (Jelicic & Petrovacki, 2009). Using the definitions of the FOCPs, Frederico and Torres (2006, 2008a, b) formulated a Noether-type theorem in the general context of the fractional optimal control in the sense of Caputo fractional derivatives. Agrawal (2008b) considered a one dimensional distributed system and used the eigenfunction approach to solve the FOC problem. The eigenfunction expansion-based scheme was also used in (Ozdemir *et al.*, 2009a) to formulate FOCPs of a 2-dimensional distributed system.

In recent years, FOCPs have been addressed in polar coordinates. Ozdemir *et al.* (2009b) presented a formulation for a 2D distributed system in polar coordinates using the separation of variables method. FOCPs of a 3D distributed system were investigated in cylindrical coordinate in (Ozdemir *et al.*, 2009c). Fractional diffusion problems were discussed in polar coordinates (Ozdemir *et al.*, 2009d) and in cylinder and spherical coordinates (Povstenko, 2008; Qi & Liu, 2009); however, those works (Ozdemir *et al.*, 2009d; Povstenko, 2008; Qi & Liu, 2009) did not discuss FOCPs.

In this paper, we present a general formulation and numerical solution scheme for FOCP in cylindrical coordinates and use a hollow cylinder case as an example to demonstrate the method. An axisymmetric hollow cylinder case is considered. Fractional derivatives are defined in the Caputo sense and separation of variable method is used to decouple the equations. The eigenfunction approach is used to eliminate the space parameter and it is indicated by the combination of state and control functions. For numerical solutions, the fractional derivative differential equations are converted into Volterra-type integral equations and the time domain is discretized into several segments. The formulation derived here is used to solve for various derivative orders and the calculation converges toward the analytical solution for integer order problems as the order approaches 1.

2. FORMULATION OF A FRACTIONAL OPTIMAL CONTROL PROBLEM

We define an FOCP in terms of the left and the right Caputo fractional derivatives (CFDs) term (Tangpong & Agrawal, 2009) that are given as the follows.

The left Caputo fractional derivative (LCFD),

$$ {}_{a}^{C}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}(\frac{d}{d\tau})^{n}f(\tau)d\tau \qquad (1)$$

and the right Caputo fractional derivative (RCFD),

$$ {}_{t}^{C}D_{b}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}(\tau-t)^{n-\alpha-1}(-\frac{d}{d\tau})^{n}f(\tau)d\tau \qquad (2)$$

where $f\ (t)$ is a time dependent function and α is the order of the derivative in the range $n-1<\alpha<n$. When α is an integer, the left (forward) and the right (backward) derivatives are replaced with D and $-D$, respectively, where D is the differential operator. Note that in the literature, the CFD generally means the LCFD.

Using the above definitions, the FOCP under consideration can be defined as follows. Find the optimal control $f(t)$ that minimizes the performance index

$$J(f) = \int_{0}^{1}F(w,f,t)dt \qquad (3)$$

subject to the dynamic constraints

$$ {}_{0}^{C}D_{t}^{\alpha}w = G(w,f,t) \qquad (4)$$

and the initial conditions

$$w(0) = w_{0} \qquad (5)$$

where $w(t)$ and $f(t)$ are the state and control variables, respectively, F and G are two arbitrary functions, and w_0

represents the initial condition of the state variable. Note that Eq. (3) may also include additional terms containing state variable at the end points. When $\alpha = 1$, the above problem reduces to a standard optimal control problem. Here the limits of the integration are taken as 0 and 1 considering a normalized case. Furthermore, we consider $0 < \alpha < 1$. An end point term can also be included in the performance index and any integration limits can be considered with any order of the derivative. The conditions considered here are for simplicity only.

To obtain the necessary equations, we combine Eqs. (3) and (4) using a Lagrange multiplier technique, and then take the variations of resulting equation and apply integration by parts to modify the equation so that it does not contain variations of a derivative term. After imposing necessary terminal conditions and setting the coefficients of $\delta\lambda, \delta w,$ and δf to zero, the following equations are obtained:

$$ {}_0^C D_t^\alpha w = G(w, f, t), \tag{6} $$

$$ {}_t^C D_1^\alpha \lambda = \frac{\delta F}{\delta w} + \left(\frac{\delta G}{\delta w}\right)^T \lambda, \tag{7} $$

$$ \frac{\delta F}{\delta f} + \left(\frac{\delta G}{\delta f}\right)^T \lambda = 0 \,, \tag{8} $$

$$ w(0) = w_0 \text{ and } \lambda(1) = 0 \,, \tag{9} $$

where λ is the Lagrange multiplier also known as the co-state or adjoint variable. The details of the derivations of Eqs. (6)-(9) are given in (Agrawal, 2004).

Equations (6)–(8) represent the Euler–Lagrange equations for the FOCP. These equations give the necessary conditions for the optimality of the FOCP considered here. They are very similar to the Euler–Lagrange Equations for classical optimal control problems, except that the resulting differential equations contain the left and the right fractional derivatives. Observe that Eq. (6) contains the LCFD whereas Eq. (7) contains the RCFD. This clearly indicates that the solution of such optimal control problems requires knowledge of not only forward derivatives but also of backward derivatives to account for all end conditions. In classical optimal control theories, such issue is either not discussed or not clearly stated largely because the backward derivative of order 1 turns to be the negative of the forward derivative of order 1. It can be demonstrated that in the limit of $\alpha \to 1$, Eqs. (6)–(8) reduce to those obtained using the standard methods for classical optimal control problems.

In the next section, we present a formulation of FOC of a distributed system and use the eigenfunction method to reduce the formulation into a set of fractional differential equations and each equation can be solved independently of the others.

3. FORMULATION OF FOC OF A HOLLOW CYLINDER WITH AXIAL SYMMETRY

In this section, we present formulation of FOC of a hollow cylinder with axial symmetry. Let us find the control $f(r, z, t)$ that minimizes the cost functional

$$ J(f) = \frac{1}{2}\int_0^1 \int_0^L \int_0^{2\pi} \int_0^R [Q' w^2(r, z, \theta, t) + R' f^2(r, z, \theta, t)] r dr d\theta dz dt \tag{10} $$

subject to the system dynamic constraints

$$ \frac{\partial^\alpha w}{\partial t^\alpha} = \beta\left(\frac{\partial^2 w(r,z,\theta,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,z,\theta,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,z,\theta,t)}{\partial \theta^2} + \frac{\partial^2 w(r,z,\theta,t)}{\partial z^2}\right) + f(r, z, \theta, t). \tag{11} $$

For an axisymmetric case, there is no variations in θ, and therefore, Eqs. (10) and (11) become

$$ J(f) = \frac{1}{2}\int_0^1 \int_0^R \int_0^L [Q' w^2(r, z, t) + R' f^2(r, z, t)] r dr dz dt \tag{12} $$

$$ \frac{\partial^\alpha w}{\partial t^\alpha} = \beta\left(\frac{\partial^2 w(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,z,t)}{\partial r} + \frac{\partial^2 w(r,z,t)}{\partial z^2}\right) + f(r, z, t). \tag{13} $$

The initial condition is

$$ w(r, z, 0) = w_0(r, z), \tag{14} $$

and the boundary condition is

$$ w(a, z, t) = w(R, z, t) = 0, \quad t > 0, \tag{15} $$

where $w(r, z, t)$ and $f(r, z, t)$ are the state and control functions that depend on radius r, length z and time t. $\frac{\partial^\alpha w}{\partial t^\alpha}$ is the partial Caputo derivatives of $w(r, z, t)$ where α is the order of derivative. Here we consider $0 < \alpha < 1$. Q' and R' are the two arbitrary functions that may depend on time. R and a are respectively the outer radius and inner radius of the cylinder, and L is the cylinder's length. For convenience, the upper limit of time t is taken as 1.

The eigenfunction approach is used here to decouple the equations. The state function $w(r, z, t)$ and the control function $f(r, z, t)$ are found to be

$$ w(r, z, t) = \sum_{i=0}^n \sum_{j=1}^m q_{ij}(t)\, u_0(\lambda_j r) sin(i\pi \tfrac{z}{L}) \,, \tag{16} $$

$$ f(r, z, t) = \sum_{i=0}^n \sum_{j=1}^m p_{ij}(t)\, u_0(\lambda_j r) sin(i\pi \tfrac{z}{L}) \,, \tag{17} $$

where

$$ u_0(\lambda_j r) = Y_0(\lambda_j \mathrm{a}) J_0(\lambda_j r) - J_0(\lambda_j \mathrm{a}) Y_0(\lambda_j r) \tag{18} $$

are the eigenfunctions in the radial direction, and $sin\left(i\pi \frac{z}{L}\right)$ are the eigenfunctions in the axial direction. J_0 and Y_0 are the zero-order Bessel function of the first kind and the second kind, respectively, and λ_j are the roots of the characteristic equation for the eigenfunctions in the radial direction.

$q_{ij}(t)$ and $p_{ij}(t)$ are the state and control eigencoordinates. Substituting Eqs. (16) and (17) into (12), we obtain the cost function

$$J = \frac{L}{4}\int_0^1 \sum_{i=0}^{n}\sum_{j=1}^{m}\left(Q' {q_{ij}}^2 + R' {p_{ij}}^2\right)\left(\int_a^R [u_0(\lambda_j r)]^2 r dr\right)dt \; . \tag{19}$$

By substituting Eqs. (16) and (17) into Eq. (13) and equating the coefficients of $u_0(\lambda_j r) sin(i\pi \frac{z}{L})$, we obtain

$${}_0^C D_t^\alpha q_{ij}(t) = -\beta\left(\lambda_j^2 + \left(i\pi \frac{z}{L}\right)^2\right) q_{ij}(t) + p_{ij}(t). \tag{20}$$

From Eqs. (6)-(9), (19) and (20), we obtain

$${}_t^C D_1^\alpha p_{ij}(t) = -\frac{Q'}{R'} q_{ij}(t) - \beta\left(\lambda_j^2 + \left(i\pi \frac{z}{L}\right)^2\right) p_{ij}. \tag{21}$$

Substituting Eq. (16) into Eq. (14), and then multiplying the equation by $ru_0(\lambda_j r)$ on both sides and integrate from a to R, we find the initial condition of the eigencoordinates

$$q_{ij}(0) = \frac{2\int_0^L\left(\int_a^R r w_0(r) u_0(\lambda_j r) dr\right) Sin(i\pi \frac{z}{L})}{L\int_a^b r[u_0(\lambda_j r)]^2 dr} \, . \tag{22}$$

Equations (20) and (21) have $j+1$ sets of decoupled equations that can be solved separately. A numerical scheme that can be used to solve Eqs. (20) and (21) is given in the following section.

For $\alpha = 1$, Eqs. (20) and (21) reduce to

$$\dot{q_{ij}}(t) = -\beta\left(\lambda_j^2 + \left(i\pi \frac{z}{L}\right)^2\right) q_{ij}(t) + p_{ij}(t) \quad , \tag{23}$$

$$\dot{q_{ij}}(t) = -\frac{Q'}{R'} q_{ij}(t) - \beta\left(\lambda_j^2 + \left(i\pi \frac{z}{L}\right)^2\right) p_{ij}. \tag{24}$$

Equations (23) and (24) represent a set of linear differential equations and the closed form solutions are given in (Agrawal, 2008b).

4. NUMERICAL ALGORITHM

This section briefly describes the numerical algorithm for the FOCPs, similar to that presented in (Tangpong & Agrawal, 2009; Agrawal, 2006). For simplicity in the discussions to follow, we consider the following generic form to represent the FOCPs:

$${}_0^C D_t^\alpha w = -\mathrm{A}w + \mathrm{B}f, \tag{25}$$

$${}_t^C D_1^\alpha f = -\mathrm{C}w - \mathrm{D}f, \tag{26}$$

$$w(0) = w_0 \tag{27}$$

$$\text{and } f(1) = 0. \tag{28}$$

Equations (25) and (26) can be expressed in the Volterra integral form as follows.

$$w(t) = w_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{(\alpha-1)}\left(Bf(\tau) - Aw(\tau)\right)d\tau \tag{29}$$

$$f(t) = -\frac{1}{\Gamma(\alpha)}\int_t^1 (\tau - t)^{(\alpha-1)}\left(Df(\tau) + Cw(\tau)\right)d\tau. \tag{30}$$

To develop the numerical algorithm, we divide the time domain [0, 1] into N equal intervals, and number the nodes from 0 to N. Here N is a positive integer. The time at node j is given as $t_j = jh$, $j = 0,\ldots,N$ and $h = 1/N$. Furthermore, we approximate $w(t)$ and $f(t)$ between two successive temporal nodes linearly. Using the above definitions and approximations, Eq. (29) reduces to (Tangpong & Agrawal, 2009)

$$w(t) = w_0 - A\sum_{j=0}^{i} a_{ij}\, w(t_j) + B\sum_{j=0}^{i} a_{ij}\, f(t_j), \quad i = 1,2,\ldots,N \tag{31}$$

where the coefficients a_{ij} are defined as

$$a_{ij} = d_1\begin{cases}(i-1)^\beta - i^\beta + \beta i^\alpha & if\ j = 0\\ (k+1)^\beta + (k-1)^\beta - 2k^\beta & if\ 1 \le j \le i-1\\ 1 & if\ j = i\end{cases} \tag{32}$$

Here $d_1 = h^\alpha/\Gamma(\alpha+2)$, $\beta = \alpha + 1$ and $k = i - j$. Following the same approach, the value of $f(t)$ at node i becomes

$$f(t_i) = -C\sum_{j=i}^{N} b_{ij}\, w(t_j) - D\sum_{j=i}^{N} b_{ij}\, f(t_j), \quad i = 0,1,\ldots,N-1, \tag{33}$$

where

$$b_{ij} = d_1\begin{cases}1 & if\ j = i\\ (k+1)^\beta + (k-1)^\beta - 2k^\beta & if\ i+1 \le j \le N-1\\ (M-1)^\beta - M^\beta + \beta M^\alpha & if\ j = N\end{cases} \tag{34}$$

Here $M = N - i$ and $k = j - i$. Equations (31) and (33) represent a set of $2N$ linear algebraic equations in terms of $2N$ unknowns, which can be solved using a standard linear algebraic equations solver.

5. NUMERICAL RESULTS AND DISCUSSIONS

This section presents simulation results for the FOC of a hollow cylinder with axial symmetry. The initial condition was taken as

$$w_0(r,z) = (r-a)(r-R)\sin(2\pi \frac{z}{L}) \tag{35}$$

For simplicity, we further considered $Q' = R' = L = 1, a = 0.5$ and $R = 1$. For simulation purposes, we discretized the spatial dimensions and the time domain into several segments and took different values of α. We first conducted convergence studies on the number of eigenfunctions in both the radial and axial directions, and found that the results converged with $m = 3$ and $n = 5$, where m is the number of eigenfunctions in the radial direction and n is the number of eigenfunctions in the axial direction. All simulation results presented in this section were based on these values of m and n.

Figures 1 and 2 demonstrate the state and control variables as a function of time, and they both converge as the time steps are reduced. For each problem, the convergence studies of the number of eigenfunctions and time steps need to be conducted first before other parameter studies, and the convergence criterion can vary with the specific problem. Figures 3 and 4 show changes of the state and control variables as functions of time for various orders of α and also compare the numerical result with the analytical result when $\alpha = 1$. In the limit of $\alpha = 1$, the numerical solution recovers the analytical solution of the integer order optimal control problem. The agreement of analytical results with the numerical results when $\alpha = 1$ shows that the numerical algorithm is accurate.

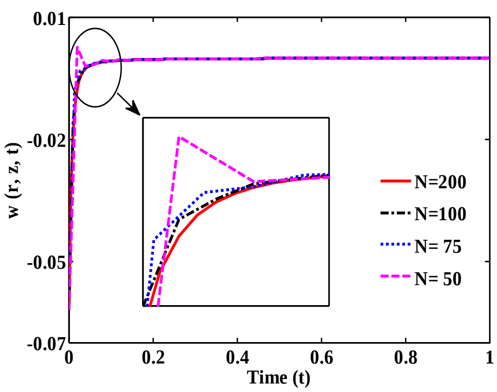

Fig. 1. Convergence of the state variable $w(r = 0.75, z = 0.25, t)$ for different number of time segments for $\alpha = 0.90$.

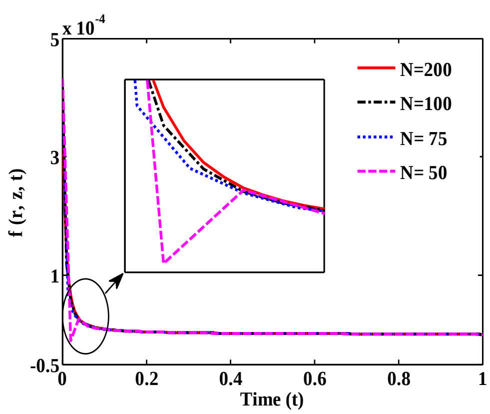

Fig. 2. Convergence of the control variable $f(r = 0.75, z = 0.25, t)$ for different number of time segments for $\alpha = 0.90$.

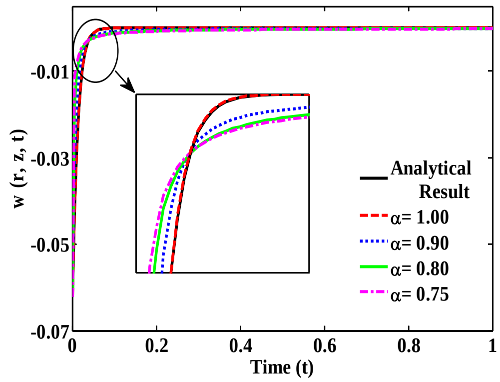

Fig. 3. State variable $w(r = 0.75, z = 0.25, t)$ for different values of α with $N = 200$.

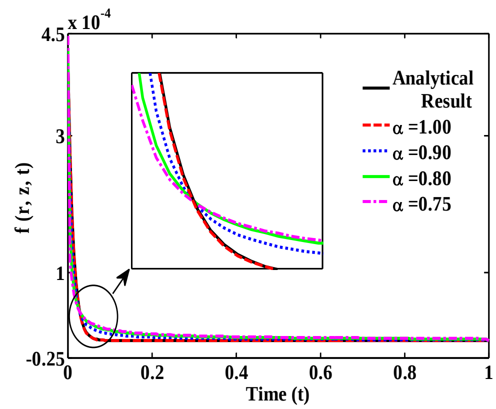

Fig. 4. Control variable $f(r = 0.75, z = 0.25, t)$ for different values of α with $N = 200$.

Figures 5 and 6 are the surface plots of the state and control variables in the radial direction. In both figures, the state and control variables initially have different values across the radial dimension due to the initial conditions; as the time progresses, each variable reaches the same value across the radius. The phenomenon shown in Fig. 5 is typical of a diffusion process.

Figures 7 and 8 are the three dimensional responses of the hollow cylinder in longitudinal direction. Similar to the phenomena shown in Figs. 5 and 6, the state and control variables each approaches the same value across the length as the time progresses, representing a diffusion process. The dynamics constraint equation (13) becomes a heat diffusion equation when $\alpha = 1$; when $\alpha = 0.9$, the dynamics governed by Eq. (13) is close to a diffusion process, but not exactly the same as the integer order derivative case. For such dynamic problems, the fractional order differential equation can give more accurate results than the integer order differential equation.

The results discussed above are representative, and similar trends are also observed for other values of m, n, N and α.

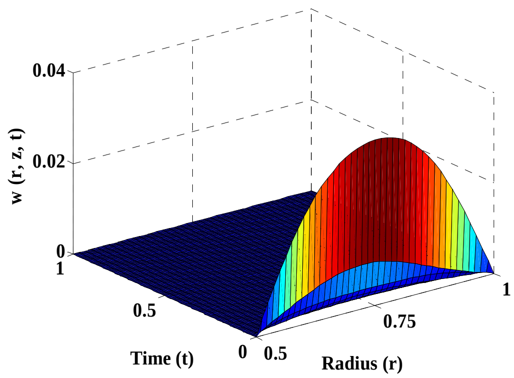

Fig. 5. State variable $w\,(r, z = 0.9, t)$ for $N = 100$ and $\alpha = 0.90$.

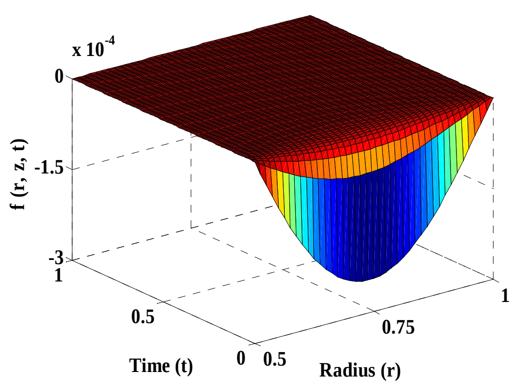

Fig. 6. Control variable $f\,(r, z = 0.9, t)$ for $N = 100$ and $\alpha = 0.90$.

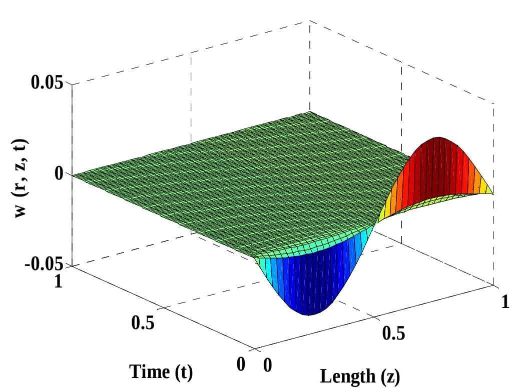

Fig. 7. State variable $w(r = 0.9, z, t)$ for $N = 100$ and $\alpha = 0.90$.

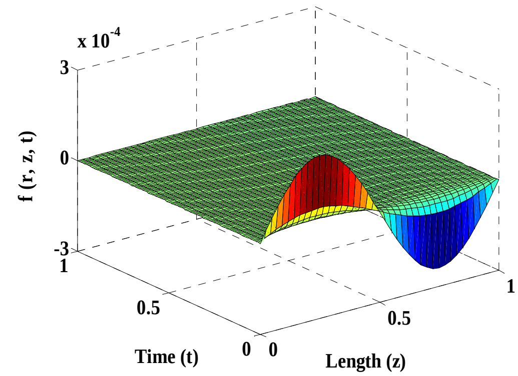

Fig.8. Control variable $f(r = 0.9, z, t)$ for $N = 100$ and $\alpha = 0.90$.

6. CONCLUSIONS

A general formulation and a numerical scheme for FOC of a distributed system in cylindrical coordinate system are presented. Naturally, an axisymmetric problem arises for such a problem and a hollow cylinder having axial symmetry is discussed as the example. Partial fractional time derivatives are defined in the Caputo sense and the performance index of the FOCP is defined as a function of both state and control variables. The separation of variable method and the eigenfunction approach are used to decouple the equations and define the problem in terms of the state and control variables. For numerical calculations, the fractional differential equations are expressed in Volterra integral form. Several numerical simulations are discussed including convergence studies of the state and control variables with respect to the number of segments in the time domain, and the convergence of the number of eigenfunctions in the radial direction, as well as in the axial direction. The numerical results of the state and the control variables recover the analytical results as the order α approaches 1. The three dimensional plots of the state and control variables are also generated that clearly show a diffusion process in the cylinder.

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