Generalized GL, Caputo, and Riemann-Liouville derivatives for analytic functions

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Abstract: The formulations of Riemann-Liouville and Caputo derivatives in the complex plane are presented. Two versions corresponding to the whole or half plane. It is shown that they can be obtained from the Grünwald-Letnikov derivative.

Keywords: Grünwald-Letnikov, Riemann-Liouville, Caputo Cauchy.

1. INTRODUCTION

The most known and popular fractional derivatives are almost surely the Riemann-Liouville (RL) and the Caputo (C) derivatives (Kilbas et al, 2006, Podlubny, 1999; Samko et al, 1987). Without considering the reserves put before (Ortigueira et al, 2005), we are going to face two related questions:

1) Can we formulate those derivatives in the complex plane?
2) Is there a coherent relation between those derivatives and the incremental ratio based Grünwald-Letnikov (GL) or its generalization?

In this paper we are going to give positive answers for those questions, by constructing formulations in the complex plane obtained from the GL. We are going to follow the procedures previously (Ortigueira and Coito,2004; Ortigueira,2006) . We start from the general formulation of the incremental ratio based Grünwald-Letnikov (Ortigueira and Trujillo, 2009) and consider only the forward case. From it we obtain an integral formulation for the difference that leads to the Cauchy type fractional derivative {see, for example, (Campos, 1984,1987; Nishimoto,1989;Samko et al,1987)}. This is the main contribution of this paper: establish a bridge between the GL derivative and integral formulations in the sense that these come logically from the other. The particularization for the Hankel integration path allows us to obtain regularised integrals for the positive order case. This gives a logical justification for the previous derivation in the Caputo derivative case and the pos derivation in the Riemann-Liouville case.

The paper outlines as follows. In section 2 we will present the general formulation of the GL derivative, while the integral formulation is obtained in section 3. Using the Hankel path we obtain the corresponding regularised derivative, presented in section 4. We consider two cases: functions analytic on the whole complex plane and functions defined on a half plane and identically null on the other.

Remark: In this paper we deal with a multivalued expression z^α. As is well known, to define a function we have to fix a branch cut line and choose a branch (Riemann surface). It is a common procedure to choose the negative real half-axis as branch cut line. Unless stated the contrary, in what follows we will assume that we adopt the principal branch and assume that the obtained function is continuous above the branch cut line. With this, we will write $(-1)^\alpha = e^{j\alpha\pi}$.

2. THE GL FRACTIONAL DERIVATIVE

The Grünwald-Letnikov fractional derivative is the most straightforward definition, since it is the generalization of the classic derivative that is recovered when the order becomes positive integer. We here introduce the following modification of the mentioned fractional derivative by the limit of the fractional incremental ratio (Ortigueira and Trujillo, 2009)

$$D_\theta^\alpha f(z) = e^{-j\theta\alpha} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z - kh)}{|h|^\alpha} \tag{1}$$

where $\binom{\alpha}{k}$ stands for the binomial coefficients and $h = |h|e^{j\theta}$ is a complex number, with $\theta\in(-\pi,\pi]$. The above definition is valid for any order, real or complex. In general, if $\theta = 0$, we call (1) the forward Grünwald-Letnikov derivative

$$D_f^\alpha f(z) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z - kh)}{h^\alpha} \tag{2}$$

Let $\theta = \pi$ and substitute h for |h| to obtain the backward Grünwald-Letnikov derivative

$$D_b^{\alpha} f(z) = \lim_{h \to 0+} e^{-j\pi\alpha} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^{\alpha}} \quad (3)$$

It is important to enhance again that when α is a positive integer we obtain the classic expressions for the integer order derivatives. It is not a simple task to formulate the weakest conditions that ensure the existence of the fractional derivatives (1), (2) and (3), although we can give some necessary conditions for their existence. To study the existence conditions for the fractional derivatives we must care about the behaviour of the function along the half straight-line $z \pm nh$ with $n \in Z^+$. If the function is zero for Re(z) < a∈R (resp. Re(z) > a) the forward (backward) derivative exists at every finite point of f(z). In the general case, we must have in mind the behavior of the binomial coefficients. They verify $\left|\binom{\alpha}{k}\right| \le \frac{A}{k^{\alpha+1}}$ meaning that $f(z).\frac{A}{k^{\alpha+1}}$ must decrease, at least as $\frac{A}{k^{|\alpha|+1}}$ when k goes to infinite. For example considering the forward case, if $\alpha > 0$, it is enough that f(z) be bounded in the left half plane, but if $\alpha < 0$, f(z) must decrease to zero to obtain a convergent series. In particular, this suggests that Re(h) > 0 and Re(h) < 0 should be adopted for right and left functions ([1]), respectively in agreement with Liouville reasoning (Dugowson). In particular, they should be used for the functions such that f(z)=0 for Re(z)<0 and f(z)=0 for Re(z)>0, respectively[2]. This is very interesting, since we conclude that the existence of the fractional derivative depends only on what happens in one half complex plane, left or right.

3. THE GENERAL CAUCHY DERIVATIVE

Assuming that f(z) is analytical in a region that includes the straight line in figure 1, we can write

$$D_\theta^{\alpha} f(z) = \lim_{|h| \to 0} \frac{\Gamma(\alpha+1)}{2\pi jh} \int_{C_d} f(w) \frac{\Gamma(\frac{w-z}{h})}{\Gamma(\frac{w-z}{h}+\alpha+1)} dw \quad (4)$$

where C_d is any U shaped contour that encircles the half straight line starting at z. Commuting the limiting and integral operations and computing the limit (Ortigueira and Coito,2004), we obtain the generalised Cauchy integral

$$D_\theta^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi j} \int_{C_d} f(w) \frac{1}{(w-z)^{\alpha+1}} dw \quad (5)$$

where C_d is any U shaped contour that encircles the half straight line starting at z that is the branch cut line of $w^{-\alpha-1}$. For existence conditions see (Campos,1984,1993).

[1] We say that *f*(z) is a right [left] function if f(-∞) = 0 [*f*(+∞) = 0].

[2] By breach of language we call them causal and anti-causal functions borrowing the system terminology.

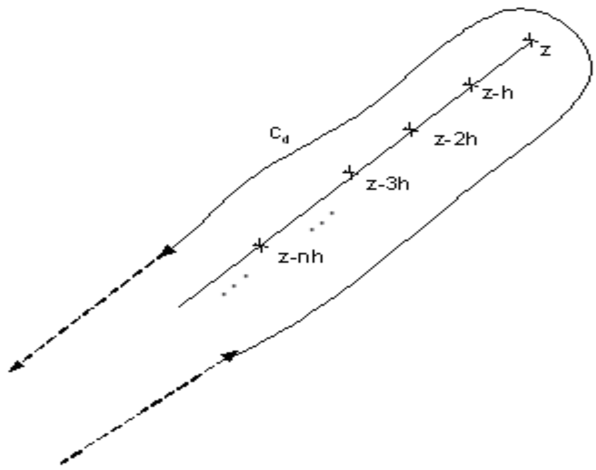

Fig. 1 – integration paths and poles for the integral representation of fractional order differences

4. THE REGULARISED DERIVATIVE

Consider the generalised Cauchy formula (5) and rewrite it in a more convenient format obtained by a simple translation:

$$D_\theta^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi j} \int_{C} f(w+z) \frac{1}{w^{\alpha+1}} dw \quad (6)$$

where we assume that f(z) is analytic in a region that contains the contour C. This is the translated version of C_d. Here we will choose C as a special integration path: the Hankel contour represented in figure 2. We assume that it surrounds the selected branch cut line. This is described by $x.e^{j(\theta+\pi)}$, with $x \in R^+$ and $\theta \in [-\pi,\pi)$. The circle has a radius equal to ρ small enough to allow it to stay inside the region of analyticity of f(z). With this contour, we can decompose (6) into three integrals along the two half-straight lines and the circle. It is interesting to remark that if α is a positive integer, the integrals along the straight lines cancel out and it remains the integral over the circle: <u>we obtain the usual Cauchy formula</u>.

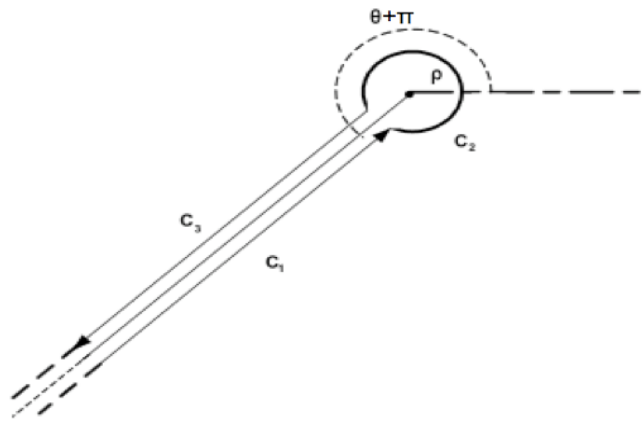

Figure 2 – The Hankel contour used in computing the derivative defined in equation (5)

If α is a negative integer, the integral along the circle is zero and we are led to the well known repeated integration formula (Nishimoto,1989; Samko et al 1987). In the general α case we need the two terms. Performing the computations, we obtain the general regularised derivative

$$D_\theta^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} e^{-j\theta\alpha} \int_{\gamma_\theta} \frac{\left[f(z-w) - \sum_0^N \frac{f^{(n)}(z)\,(-1)^n}{n!} w^n \right]}{w^{\alpha+1}} dw \qquad (7)$$

where γ_θ is a half straight line starting at w=0 (Ortigueira, 2003) and $N=\lfloor\alpha\rfloor$ ([3]). If α is negative, the summation inside the integral in (7) is zero. We must refer that this derivative formulation does verify the additivity law of the orders (8) contrarily to the stated by Li et al (2009).

5. RL AND C DERIVATIVES IN THE COMPLEX PLANE

As it is easy to prove, we have, from (1)

$$D_\theta^\alpha \left[D_\theta^\beta f(t) \right] = D_\theta^\beta \left[D_\theta^\alpha f(t) \right] = D_\theta^{\alpha+\beta} f(t) \qquad (8)$$

provided that both derivatives (of orders α and β) exist. This means that the fractional derivative as introduced in (1) verifies the semi group property in which refers to the derivative orders. This is important and not enjoyed by other derivatives. So, we can write:

$$D_\theta^\alpha f(z) = \frac{\Gamma(\alpha-\beta+1)}{2\pi j} \int_C f^{(\beta)}(w+z) \frac{1}{w^{\alpha-\beta+1}} dw \qquad (9)$$

Let us choose, $\beta=n\in Z^+$ and $\varepsilon=n-\alpha>0$. We obtain:

$$D_\theta^\alpha f(z) = \frac{\Gamma(-\varepsilon+1)}{2\pi j} \int_C f^{(n)}(w+z)\, w^{\varepsilon-1} dw \qquad (10)$$

or

$$D_\theta^\alpha f(z) = \frac{\Gamma(-\varepsilon+1)}{2\pi j} \int_{C_d} f^{(n)}(w)\,(w-z)^{\varepsilon-1} dw \qquad (11)$$

that can be considered as a Caputo-Cauchy derivative, provided the integral exists. This is valid because f(z) is analytic and we assumed that the GL derivative exists. Consider again the integration path in figure 2. As before, we can decompose (10) into three integrals along the two half-straight lines and the circle. We have, then:

$$D_\theta^\alpha f(z) = \frac{\Gamma(-\varepsilon+1)}{2\pi j} \left[\int_{C_1} + \int_{C_2} + \int_{C_3} \right] f^{(n)}(w+z)\, w^{\varepsilon-1} dw \qquad (12)$$

[3] $\lfloor\alpha\rfloor$ means "the greatest integer less than or equal to α".

Over C_1 we have $w=x.e^{j(\theta-\pi)}$, while over C_3 we have $w=x.e^{j(\theta+\pi)}$, with $x\in R^+$, over C_2 we have $w=\rho e^{j\varphi}$, with $\varphi\in(\theta-\pi,\theta+\pi)$. We can write, at last:

$$D_\theta^\alpha f(z) = \frac{\Gamma(\varepsilon-1)}{2\pi j} \int_\infty^\rho f^{(n)}(-x.e^{j\theta}+z)\, e^{j\varepsilon(\theta-\pi)} x^{\varepsilon-1} dx + $$

$$+ \frac{\Gamma(\varepsilon-1)}{2\pi j} \int_\rho^\infty f^{(n)}(-x.e^{j\theta}+z)\, e^{j\varepsilon(\theta+\pi)} x^{\varepsilon-1} dx +$$

$$+ \frac{\Gamma(-\varepsilon+1)}{2\pi j} \rho^\varepsilon \int_{\theta-\pi}^{\theta+\pi} f^{(n)}(\rho.e^{j\varphi}+z)\, e^{j\varepsilon\varphi} j d\varphi \qquad (13)$$

For the first term, we have:

$$\int_\infty^\rho f^{(n)}(-x.e^{j\theta}+z)\, e^{j\varepsilon(\theta-\pi)}.x^{\varepsilon-1} dx +$$

$$+ \int_\rho^\infty f^{(n)}(-x.e^{j\theta}+z)\, e^{j\varepsilon(\theta+\pi)} x^{\varepsilon-1} dx =$$

$$= \left[-e^{j\varepsilon(\theta-2\pi)} + e^{j\varepsilon\theta} \right] \int_\rho^\infty f^{(n)}(x.e^{j\theta}+z)\, x^{\varepsilon-1} dx$$

$$= e^{j\varepsilon\theta}.\left[1 - e^{-j2\pi\varepsilon}\right] \int_\rho^\infty f^{(n)}(x.e^{j\theta}+z)\, x^{\varepsilon-1} dx$$

$$= e^{j\varepsilon\theta}.2j.\sin(\varepsilon\pi) \int_\rho^\infty f^{(n)}(-x.e^{j\theta}+z)\, x^{\varepsilon-1} dx \qquad (14)$$

where we assumed that $f^{(n)}(x.e^{j(\theta-\pi)}+z) = f^{(n)}(x.e^{j(\theta+\pi)}+z)$, because f(z) is analytic. For the second term, we begin by noting that the analyticity of the function f(z) allows us to write:

$$f(-x.e^{j\theta}+z) = \sum_0^\infty \frac{f^{(n)}(z)}{n!} (-1)^{-n} x^n e^{jn\theta} \qquad (15)$$

for $x < r\in R^+$. We have also

$$f^{(n)}(-x.e^{j\theta}+z) = \sum_n^\infty \frac{(-k)_n f^{(k)}(z)}{k!} (-x)^{k-n} e^{jk\theta} \qquad (16)$$

and then:

$$j\rho^\varepsilon \int_{\theta-\pi}^{\theta+\pi} f^{(n)}(\rho.e^{j\varphi}+z)\, e^{j\varepsilon\varphi} d\varphi =$$

$$= j\rho^\varepsilon \sum_n^\infty \frac{(-1)^n(-k)_n f^{(k)}(z)}{k!} (-1)^k \rho^{k-n} \int_{\theta-\pi}^{\theta+\pi} e^{j(k+\varepsilon)\varphi} d\varphi \qquad (17)$$

Performing the integration, we have:

$$j\rho^{\varepsilon}\int_{\theta-\pi}^{\theta+\pi} f^{(n)}(\rho.e^{j\varphi}+z)\, e^{j\varepsilon\varphi}\, d\varphi =$$

$$= 2j.\, e^{j\varepsilon\theta}\sin(\varepsilon\pi)\sum_{n}^{\infty}\frac{(-1)^n(-k)_n f^{(k)}(z)}{k!}(-1)^k\frac{e^{jk\theta}\rho^{k-n+\varepsilon}}{(k+\varepsilon)} \qquad (18)$$

As ρ decreases to zero, the summation in the last expression goes to zero. This means that when $\rho \to 0$

$$D^{\alpha}_{\theta} f(z) = e^{j\varepsilon\theta}\cdot\frac{1}{\Gamma(\varepsilon)}\int_0^{\infty} f^{(n)}(-x.e^{j\theta}+z)\, x^{\varepsilon-1}\, dx$$

This can be considered as a generalised Caputo derivative. In fact, with $\theta = 0$, we obtain:

$$D^{\alpha}_{f} f(z) = \frac{1}{\Gamma(\varepsilon)}\int_0^{\infty} f^{(n)}(z-x)\, x^{\varepsilon-1}\, dx =$$

$$= \frac{1}{\Gamma(\varepsilon)}\int_{-\infty}^{z} f^{(n)}(\tau)\,(z-\tau)^{\varepsilon-1}\, d\tau$$

that is the forward Caputo derivative in R.

Now, return to (5) and put $\alpha = n-\varepsilon$, with $\varepsilon > 0$, again:

$$D^{\alpha}_{\theta} f(z) = \frac{\Gamma(\alpha+1)}{2\pi j}\int_{C_d} f(w)\frac{1}{(w-z)^{n-\varepsilon+1}}\, dw$$

$$= \frac{\Gamma(\alpha+1)}{2\pi j}\int_{C} f(w)(w-z)^{\varepsilon-n-1}\, dw \qquad (19)$$

But, as

$$(w-z)^{\varepsilon-n-1} = \frac{1}{(1-\varepsilon)_n} D^n_z (w-z)^{\varepsilon-1} \qquad (20)$$

we obtain, by commuting the operations of derivative and integration

$$D^{\alpha}_{\theta} f(z) = D^n\left[\frac{\Gamma(-\varepsilon+1)}{2\pi j}\int_{C} f(w)\,(w-z)^{\varepsilon-1}\, dw\right] \qquad (21)$$

We may wander about the validity of the above commutation. We remark that the resulting integrand function is well behaved than the original, ensuring that we gain something on doing such operation. The formula (21) is the complex version of the Riemann-Liouville derivative that we can write in the format

$$D^{\alpha}_{\theta} f(z) = D^n\left[\frac{\Gamma(-\varepsilon+1)}{2\pi j}\int_{C} f(w+z)\, w^{\varepsilon-1}\, dw\right] \qquad (22)$$

Using again the Hankel integration path, we obtain easily:

$$D^{\alpha}_{\theta} f(z) = e^{j\varepsilon\theta} D^n\left[\frac{1}{\Gamma(\varepsilon)}\int_0^{\infty} f(-x.e^{j\theta}+z)\, x^{\varepsilon-1}\, dx\right] \qquad (23)$$

that is a generalised RL derivative. With $\theta = 0$, we obtain the usual formulation of the RL in R. With $\theta = \pi$, we obtain aside a factor the "right" RL derivative.

6. HALF PLANE DERIVATIVES

Let us assume that $f(z) = 0$ for $Re(z) < 0$. In this case, the summation in (1) goes only to $\lfloor Re(z)/Re(h)\rfloor$ and the integration path in (4) is finite, closed and completely in the right half complex plane. In figure 3 we assumed that z and h are real.

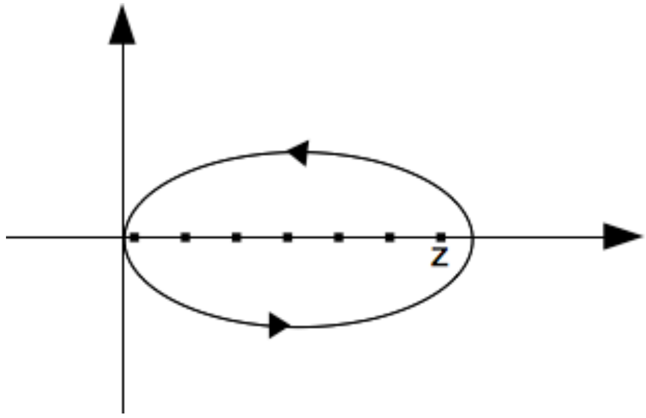

Figure 3 – The contour used in computing the half plane derivatives

Consider a sequence h_n going to zero. The number of poles inside the integration path is $\lfloor Re(z)/Re(h)\rfloor$, but in the limit, the quotient of 2 gamma functions will give rise to a multivalued expression that forces us to insert a branch cut line that starts at z and ends at $-\infty$. Over this line the integrand is not continuous. So, we obtain:

$$D^{\alpha}_{\theta} f(z) = \frac{\Gamma(\alpha+1)}{2\pi j}\int_{C} f(w)\frac{1}{(w-z)^{\alpha+1}}\, dw +$$

$$+ \frac{\Gamma(\alpha+1)}{2\pi j}\int_{\gamma} f(w)\frac{1}{(w-z)^{\alpha+1}}\, dw \qquad (24)$$

where C is an open contour that encircles the branch cut line and γ is a small line passing at $w = 0$ whose length we will reduce to zero. However, we prefer to use the analogue to the Hankel contour. The contour γ is a short straight line over the imaginary axis. Although the integrand is not continuous, the phase has a 2π jump, the second integral above is zero. To compute the others, we are going to do a translation to obtain an integral similar to the used above.

As before and again for reducing steps, we will assume already that the straight lines are infinitely near to each other. We have, then:

$$D^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi j}\left[\int_{C_1} + \int_{C_2} + \int_{C_3}\right] f(w+z)\frac{1}{w^{\alpha+1}}\, dw \qquad (25)$$

Over C_1 we have $w = xe^{j(\theta-\pi)}$, while over C_3 we have $w = x.e^{j(\theta+\pi)}$, with $x \in R^+$, over C_2 we have $w = \rho e^{j\varphi}$, with $\varphi \in (\theta-\pi, \theta+\pi)$.

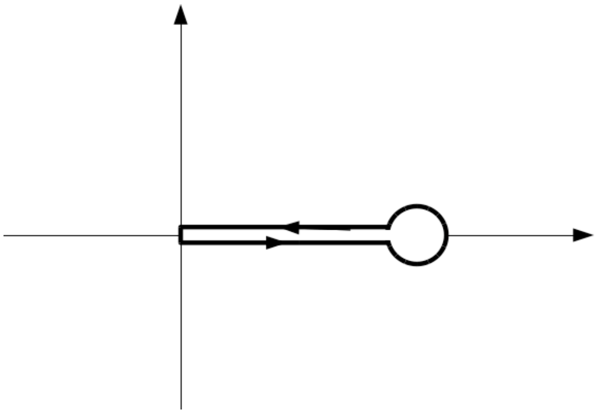

Figure 4 – The Hankel contour used in computing the derivative defined in equation (25)

Let $\zeta = |z|$. We can write, at last:

$$D_\theta^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi j}\int_\zeta^\rho f(-x.e^{j\theta} + z)\, \frac{e^{-j\alpha(\theta-\pi)}}{x^{\alpha+1}}\, dx +$$

$$+\frac{\Gamma(\alpha+1)}{2\pi j}\int_\rho^\zeta f(-x.e^{j\theta} + z)\, \frac{e^{-j\alpha(\theta+\pi)}}{x^{\alpha+1}}\, dx +$$

$$+\frac{\Gamma(\alpha+1)}{2\pi j}\,\frac{1}{\rho^\alpha}\int_{\theta-\pi}^{\theta+\pi} f(\rho.e^{j\varphi} + z)\, e^{-j\alpha\varphi}\, jd\varphi \qquad (26)$$

For the first term, we have:

$$\int_\zeta^\rho f(x.e^{j(\theta-\pi)} + z)\, \frac{e^{-j\alpha(\theta-\pi)}}{x^{\alpha+1}} dx + \int_\rho^\zeta f(-x.e^{j(\theta+\pi)} + z)\, \frac{e^{-j\alpha\theta}}{x^{\alpha+1}} dx =$$

$$= e^{-j\alpha\theta}.[e^{j\pi\alpha} - e^{-j\pi\alpha}]\int_\rho^\zeta f(-x.e^{j\theta} + z)\, \frac{1}{x^{\alpha+1}}\, dx =$$

$$= e^{-j\alpha\theta}.2j.\sin(\alpha\pi)\int_\rho^\zeta f(-x.e^{j\theta} + z)\, \frac{1}{x^{\alpha+1}}\, dx \qquad (27)$$

where we assumed again that $f(x.e^{j(\theta-\pi)} + z) = f(x.e^{j(\theta+\pi)} + z)$, because f(z) is analytic.

For the second term, we begin by noting that the analyticity of the function f(z) allows us to write:

$$f(x.e^{j\theta} + z) = \sum_0^\infty \frac{f^{(n)}(z)}{n!}\, x^n e^{jn\theta} \qquad (28)$$

for $x < r \in R^+$. We have, then:

$$j\frac{1}{\rho^\alpha}\int_{\theta-\pi}^{\theta+\pi} f(\rho.e^{j\varphi} + z)\, e^{-j\alpha\varphi}\, d\varphi =$$

$$= j\frac{1}{\rho^\alpha}\sum_0^\infty \frac{f^{(n)}(z)}{n!}\, \rho^n \int_{\theta-\pi}^{\theta+\pi} e^{j(n-\alpha)\varphi}\, d\varphi \qquad (29)$$

Performing the integration, we have:

$$j\frac{1}{\rho^\alpha}\int_{\theta-\pi}^{\theta+\pi} f(\rho.e^{j\varphi} + z)\, e^{-j\alpha\varphi}\, d\varphi =$$

$$= -\, j.e^{-j\alpha\theta}\sum_0^\infty \frac{(-1)^n f^{(n)}(z)}{n!}\, \rho^{n-\alpha} e^{jn\theta}\, \frac{2.\sin[(n-\alpha)\pi]}{(n-\alpha)} =$$

$$= -\, 2j.\, e^{-j\alpha\theta}\sin(\alpha\pi)\sum_0^\infty \frac{(-1)^n f^{(n)}(z)}{n!}\, \frac{e^{jn\theta}\rho^{n-\alpha}}{(n-\alpha)} \qquad (30)$$

But the summation in the last expression can be written in another interesting format:

$$\sum_0^\infty \frac{(-1)^n f^{(n)}(z) e^{jn\theta}}{n!}\, \frac{\rho^{n-\alpha}}{(n-\alpha)} =$$

$$= \left[-\sum_0^N \frac{(-1)^n f^{(n)}(z)}{n!} e^{jn\theta}\int_\rho^\infty x^{n-\alpha-1} dx + \sum_{N+1}^\infty \frac{(-1)^n f^{(n)}(z)}{n!}\, \frac{e^{jn\theta}\rho^{n-\alpha}}{(n-\alpha)}\right]$$

Substituting it in (30) and joining to (27) we can write:

$$D^\alpha f(z) = K.\int_\rho^\zeta \frac{\left[f(-x.e^{j\theta} + z) - \sum_0^N \frac{f^{(n)}(z)}{n!} e^{jn\theta} (-x)^n\right]}{x^{\alpha+1}}\, dx\; -$$

$$-\, K.\sum_{N+1}^\infty \frac{f^{(n)}(z)}{n!}(-1)^n \frac{\rho^{n-\alpha}}{(n-\alpha)} + \Theta \qquad (31)$$

with

$$\Theta = -\sum_0^N \frac{f^{(n)}(z)\,(-1)^n}{n!} e^{jn\theta}\int_\zeta^\infty x^{n-\alpha-1} dx = z^{-\alpha}\sum_0^N \frac{(-1)^n f^{(n)}(z)}{n!}\, \frac{z^n}{n-\alpha}$$

If $\alpha<0$, we make the inner summation equal to zero. Using the reflection formula of the gamma function we obtain for K

$$K = -\frac{\Gamma(\alpha+1)e^{-j\pi\alpha}}{\pi}\sin(\alpha\pi) = \frac{e^{-j\theta\alpha}}{\Gamma(-\alpha)} \qquad (32)$$

Now let ρ go to zero. The second term on the right hand side in (31) goes to zero and we obtain:

$$D_\theta^\alpha f(z) = K.\int_0^\zeta \frac{\left[f(-x.e^{j\theta}+z) - \sum_0^N \frac{f^{(n)}(z)}{n!} e^{jn\theta}(-x)^n\right]}{x^{\alpha+1}} dx + \zeta^{-\alpha}\sum_0^N \frac{(-1)^n f^{(n)}(z)}{n!}\frac{z^n}{n-\alpha} \quad (33)$$

This result shows that in this situation and with $\alpha>0$ we have a regularised integral and an additional term. This means that it is somehow difficult to compute the fractional derivative by using (33): a simple expression obtained from (2)

$$D_\theta^\alpha f(z) = e^{-j\theta\alpha} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\lfloor \zeta/h \rfloor} (-1)^k \binom{\alpha}{k} f(z-kh)}{|h|^\alpha} \quad (34)$$

leads to a somehow complicated formation in (33). However, if $\alpha < 0$ we obtain:

$$D_\theta^\alpha f(z) = K.\int_0^\zeta \frac{f(-x.e^{j\theta}+z)}{x^{\alpha+1}} dx \quad (35)$$

So, we must avoid (33). To do it, remark first that, from (8) and (34) we have:

$$D_\theta^\alpha f(z) = D_\theta^n \left[D_\theta^{-\varepsilon} f(t)\right] = D_\theta^{-\varepsilon}\left[D_\theta^n f(t)\right] \quad (36)$$

This means that we can compute the α order derivative into two steps. As one step is a fractional primitivation, we avoid (33) and use (35). The order of the steps: computing the integer order derivative before or after the primitivation leads to:

$$D_\theta^\alpha f(z) = K.\int_0^\zeta \frac{f^{(n)}(-x.e^{j\theta}+z)}{x^{\alpha+1}} dx \quad (37)$$

and

$$D_\theta^\alpha f(z) = K.D^n\int_0^\zeta \frac{f(-x.e^{j\theta}+z)}{x^{\alpha+1}} dx \quad (38)$$

that are the C and RL formulations in the complex plane. However, from (36) we can write also:

$$D_\theta^\alpha f(z) = e^{-j\theta\alpha} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\lfloor \zeta/h \rfloor} (-1)^k \binom{-\varepsilon}{k} f^{(n)}(z-kh)}{|h|^\alpha} \quad (39)$$

and

$$D_\theta^\alpha f(z) = \left[e^{-j\theta\alpha} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\lfloor \zeta/h \rfloor} (-1)^k \binom{-\varepsilon}{k} f(z-kh)}{|h|^\alpha}\right]^{(n)} \quad (40)$$

These results mean that:

We can easily define C-GL (39) and RL-GL (40) derivatives

Attending to the way we followed for going from GL to C and RL, we can conclude that, in the case of analytic functions, the existence of RL or C derivatives ensure the existence of the corresponding GL. The reverse may be not correct, since the commutation of limit and integration in (6) may not be valid.

7. ACKNOWLEDGEMENT

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