Stationary states for anomalous diffusion

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Abstract: The fractional Fokker-Planck equation serves as a standard description of the anomalous diffusion. Within a current presentation we study properties of stationary states of the fractional Fokker-Planck equation in bounding potentials with special attention to the way in which stationary states are approached. It is demonstrated that the shape of the stationary state depends on the exponent characterizing the jump length distributions and the external potential. The convergence rate to the stationary state can be of the double power-law type and is determined solely by the subdiffusion parameter.

Keywords: Anomalous diffusion, fractional Fokker-Planck equation, continuous time random walks, subordination, stochastic representation.

1. INTRODUCTION

The most elaborated version of the diffusion equation corresponds to the Markovian diffusion process which usually occurs when interactions of the test particle with the surroundings are determined by a large number of independent and bounded collisions. Anomalous diffusion typically arises when we relax the assumption about the bounded type of interaction and/or when we assume that due to trapping events the waiting time distribution is of the power-law type, see Metzler and Klafter (2000). For example, assuming anomalously long waiting times ($p(t) \propto t^{-(\nu+1)}$) ($0 < \nu < 1$) and anomalously long jumps ($p(x) \propto |x|^{-(\alpha+1)}$) ($0 < \alpha < 2$), one arrives at an anomalous, non-Markovian diffusion which is described by the fractional (time and space) Fokker-Planck equation. Fractional derivatives entering the fractional diffusion equation emerge due to waiting times (fractional time derivative of the Riemann-Liouville type) and due to jump lengths (fractional space derivative of the Riesz-Weyl type). In the force free regimes, main properties of solutions to the anomalous Fokker-Planck equation are well known, see Saichev and Zaslavsky (1997); Metzler and Nonnenmacher (2002); Dybiec and Gudowska-Nowak (2009). However, the situation drastically changes when anomalous diffusion takes place in the external potential, see Dybiec (2010).

We discuss properties of the solution to the fractional Fokker-Planck equation in the single-well potentials. It is demonstrated that the existence of the stationary state is determined by the jump length distribution and the potential, see Dybiec et al. (2010). For example, if jumps are distributed according to the α-stable density, stationary states exist only for potential wells which are steep enough, see Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2007, 2010). Furthermore, if a stationary state exists, it is not of the Boltzmann-Gibbs type, see Eliazar and Klafter (2003). In particular, for the parabolic potential the stationary state is given by the same α-stable density, i.e. it has the power-law asymptotics and diverging variance, see Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2007). On the one hand, in order to produce stationary states characterized by finite variance it is necessary to consider potential wells which are steeper than parabolic. On the other hand, in the extreme case of a free particle, the stationary state does not exist. Nevertheless, it is not known what happens in the intermediate states, i.e. for potential wells of the $|x|^c$ type with $0 < c < 2$. Therefore, the current presentation investigates this problem, reporting results of recent computer simulations, see Dybiec et al. (2010).

The presence of the fractional time derivative does not affect the existence and shape of stationary states. Nevertheless, it changes the way in which the stationary states are reached, see Dybiec (2010). This can be quantified by measuring the rate of the convergence to the stationary state. We discuss how the rate of convergence can be measured and how subdiffusion influences the time dependence of solutions to the fractional Fokker-Planck equation in single-well static potentials, see Dybiec (2010). Finally, we examine the role of competition between long jumps and long waiting times on the way in which the stationary state is reached.

2. MODEL

Using the subordination method, see Magdziarz and Weron (2007); Magdziarz et al. (2008, 2007), we study properties of the solutions of the fractional Fokker-Planck equation of the form

$$\frac{\partial p(x,t)}{\partial t} = {}_0D_t^{1-\nu}\left[\frac{\partial}{\partial x}V'(x) + \frac{\partial^\alpha}{\partial|x|^\alpha}\right]p(x,t), \qquad (1)$$

see Metzler et al. (1999); Metzler and Klafter (2000). In Eq. (1), $V(x)$ represents the external static potential, $\partial^\alpha/\partial|x|^\alpha$ stands for the Riesz-Weyl fractional (space) derivative while ${}_0D_t^{1-\nu}$ denotes the Riemann-Liouville fractional (time) derivative. The Riesz-Weyl fractional derivative is defined via the Fourier transform $\mathcal{F}\left[\frac{\partial^\alpha f(x)}{\partial|x|^\alpha}\right] = -|k|^\alpha \hat{f}(x)$ while the Riemann-Liouville fractional (time) derivative is defined by the relation ${}_0D_t^{1-\nu}f(x,t) = \frac{1}{\Gamma(\nu)}\frac{d}{dt}\int_0^t dt' \frac{f(x,t')}{(t-t')^{1-\nu}}$, see Podlubny (1998); Metzler and Klafter (2000).

The fractional (space) derivative of the order α is a consequence of the power-law asymptotics of the jump length distribution, i.e. for $\alpha < 2$, $p(x) \propto |x|^{-(\alpha+1)}$. In the limit of $\alpha = 2$ the fractional derivative in Eq. (1) is replaced by the partial derivative of the second order. The fractional (time) derivative of the order ν is responsible for the non-Markovian character of the stochastic process underlying Eq. (1). For $\nu = 1$, the fractional derivative is equivalent to the identity operator and the underlying process becomes a Markovian Levy flight, see Metzler and Klafter (2000).

In following sections we report results on the existence of stationary states in single-well potentials. We also discuss the problem of the convergence rate to stationary states in the single-well quartic potential under subdiffusion.

3. RESULTS

3.1 Stationary states

The stationary solution of Eq. (1) fulfills

$$\left[\frac{\partial}{\partial x}V'(x) + \frac{\partial^\alpha}{\partial|x|^\alpha}\right]p(x) = 0. \qquad (2)$$

A stationary state of a system described by Eq. (1) is determined by the stability index α and a potential and it is the same as in the case of Lévy flights in external potentials, see Jespersen et al. (1999); Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2007). In order to confine Lévy flights the potential $V(x)$ needs to bee steep enough, see Jespersen et al. (1999); Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2007). In particular, for the parabolic potential, $V(x) = x^2/2$, the stationary state is given by the α-stable density characterized by the same stability index α as the noise in the underlying Langevin equation, see Jespersen et al. (1999); Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2007), and the order of the fractional Riesz-Weyl derivative. Consequently, for the harmonic potential, the stationary probability density is characterized by the diverging variance, see Janicki and Weron (1994); Nolan (2010). Stationary states characterized by the finite variance exist for potential wells steeper

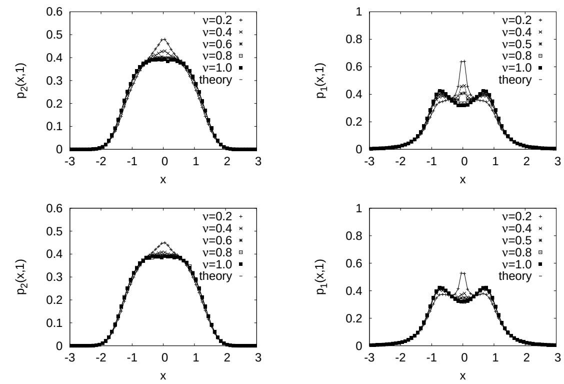

Fig. 1. Time dependent solutions of the fractional Fokker-Planck equation (1) for the stability index $\alpha = 2$ (Gaussian case, see Eq. (6), left column) and $\alpha = 1$ (Cauchy case, see Eq. (7), right column). The top row presents results at time $t = 25$, while the bottom row at time $t = 250$. Various curves correspond to different values of the subdiffusion parameter ($\nu = \{0.2, 0.4, 0.6, 0.8\}$ and $\nu = 1$). Histograms were constructed by the subordination method with the time step of integration $\Delta t = 10^{-3}$ and averaged over 10^6 realizations.

than the parabolic, see Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2007).

In general, for single-well potentials of $V(x) \propto |x|^c$ type, stationary states exist for

$$c > 2 - \alpha \qquad (3)$$

while stationary states characterized by the finite variance exist for

$$c > 4 - \alpha. \qquad (4)$$

Conditions (3), (4) and (5) can be derived from properties of the Riesz-Weyl fractional derivative, see Eq. (2) and Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2010). Furthermore, if the stationary state exists, its complementary cumulative density has asymptotic power-law decay

$$F_c(x) \propto x^{-(c+\alpha-2)} \qquad (5)$$

which is clearly demonstrated in Fig. 2, see also Dybiec et al. (2010).

Next, we focus on the investigation of the convergence rate of the system described by Eq. (1) to its stationary states, see Dybiec (2010). As a value of the stability index α we choose $\alpha = 2$ (Gaussian case) and $\alpha = 1$ (Cauchy case) because such a set of parameters leads to known stationary solutions for the single-well quartic potential ($V(x) = x^4/4$).

For $\alpha = 2$, the stationary solution of Eq. (1) is of the Boltzmann-Gibbs type and attains the form

$$p_2(x) = \frac{\sqrt{2}}{\Gamma(\frac{1}{4})}\exp\left[-\frac{x^4}{4}\right]. \qquad (6)$$

In the Cauchy ($\alpha = 1$) case the stationary solution is

$$p_1(x) = \frac{1}{\pi(x^4 - x^2 + 1)}, \qquad (7)$$

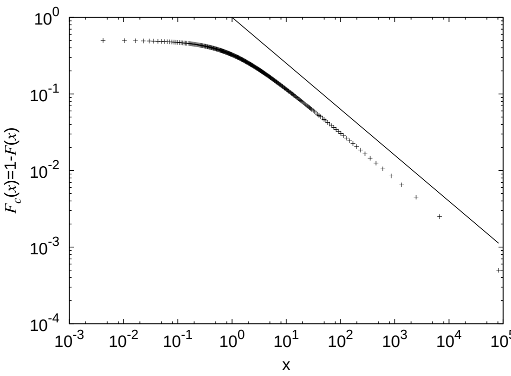

Fig. 2. The complementary cumulative density $F_c(x) = 1 - F(x) = 1 - \int_{-\infty}^{x} p(x')dx'$ for $V(x) = |x|^{1.5}$ and $\alpha = 1.1$. The solid line represents the theoretical decay predicted by Eq. (5).

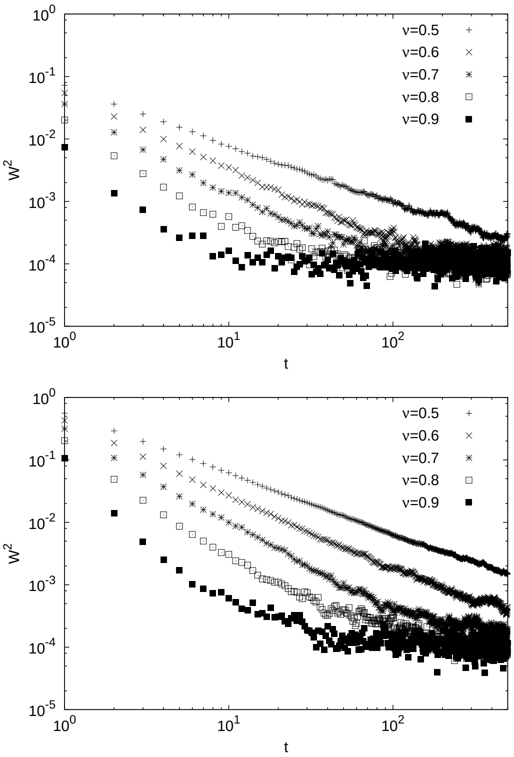

Fig. 3. The accumulated distance $W^2(t)$ (see Eq. (8)) between the estimated probability density ($\hat{p}_\alpha(x,t)$) and the stationary probability density ($p_\alpha(x)$) as a function of time for the Gaussian case (top panel, see Eq. (6)) and the Cauchy case (bottom panel, see Eq. (7)).

see Chechkin et al. (2002, 2003, 2004, 2006).

3.2 Rate of convergence to stationary states

The way in which the time dependent solution $p_\alpha(x,t)$ of Eq. (1) approaches the stationary solution $p_\alpha(x)$ can be measured by the accumulated squared difference between stationary and time dependent solutions

$$W^2(t) = \sum_{i=1}^{N} \left[p_\alpha(x_i) - \hat{p}_\alpha(x_i,t)\right]^2, \tag{8}$$

where $p_\alpha(x)$ is a theoretical stationary density, see Eqs. (6) and (7) and $\hat{p}_\alpha(x,t)$ is the numerically estimated histogram, see Dybiec (2010).

Long time asymptotic properties of $W^2(t)$ can be determined by approximating Eq. (8) as $W^2(t) = \int_{-\infty}^{\infty}[p_\alpha(x) - p_\alpha(x,t)]^2dx$ and using the time dependent solution of Eq. (1) which can be constructed by the separation of variables, see Risken (1984); Gardiner (2009); Metzler and Klafter (2000); Dybiec (2010). After the integration over the space variable x, the asymptotic behavior of the distance $W^2(t)$ can be determined by use of the asymptotic properties of the Mittag-Leffler function, see Glöckle and Nonnenmacher (1994); Metzler and Klafter (2000). Finally, one arrives at

$$W^2(t) \propto B_\alpha t^{-2\nu} + C_\alpha t^{-\nu}, \tag{9}$$

where B_α and C_α are unknown constants, see Dybiec (2010).

For $\nu < 1$ and $\alpha < 2$, Eq. (1) describes the situation when there is a competition between long waiting times and long jumps. The shape of the stationary state is determined by the potential and the exponent characterizing the jump length distribution, see Eq. (2) and Fig. 1. The subdiffusion parameter ν controls the convergence rate to the stationary state. For $\nu < 1$, the system possesses long memory about its initial condition. This is manifested by the persistent cusp at the origin which is due to the initial condition, i.e. $p(x,0) = \delta(x)$, see Fig. 1.

For $\nu < 1$, from Eq. (9) it implies that the asymptotic dependence of the distance $W^2(t)$ is of the power-law type. This is well corroborated by results of numerical simulations depicted in Fig. 3. Furthermore, for $\nu \lessapprox 1$ the crossover between initial (faster) $t^{-2\nu}$ decay and final (slower) $t^{-\nu}$ decay is visible. For $\nu = 1$, the decay of single modes of the Fokker-Planck equation is described by the exponential function. Consequently, in such a case the asymptotic decay of the distance $W^2(t)$ is exponential.

4. CONCLUSIONS

For systems described by the fractional Fokker-Planck equation the shape of stationary density is determined by the potential and the value of the exponent α characterizing the jump length distribution. Consequently, in order to assure the existence of the stationary state the potential needs to be steep enough. For single-well potentials of $|x|^c$ type, stationary states exist for $c > 2 - \alpha$, see Chechkin et al. (2002, 2003, 2004, 2006); Dybiec et al. (2007, 2010).

Using analytical arguments we have shown that the distance between time dependent solutions and stationary states can decay in time as a double-power law. Exponents characterizing the power-law decay are determined by the subdiffusion parameter ν. In the case of the Markovian diffusion, the distance between time dependent and stationary solutions displays exponential long time asymptotics.

Analytical predictions were confirmed by numerical simulations of the fractional Fokker-Planck equation which were carried out using the subordination method, see Magdziarz and Weron (2007); Magdziarz et al. (2008, 2007). Numerical simulations were performed for the quartic potential ($V(x) = x^4/4$) with Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) noise because for these two cases analytical expressions for stationary densities are known. Numerical simulations clearly showed the power-law decay of the distance between time dependent and stationary solutions. Moreover, for the value of the subdiffusion parameter close to 1, the crossover between faster and slower decay of the distance is well visible.

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