Stabilization of Fractional Order Unified Chaotic Systems via Linear State Feedback Controller

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Abstract: A unified system is presented to show three chaotic dynamics of Lorenz, Chen and Lü systems in the same structure. Theses system will be differently distinguished when the relevant parameter α is accordingly tuned. This paper deals with an asymptotically stability of fractional–order unified chaotic systems. A simple linear state feedback controller is gained to stabilize systems. This controller will be shown increases the stability region with respect to their integer order systems. Simulation results are demonstrated for Chen and Lü fractional –order systems to illustrate the effectiveness of the proposed control scheme.

Keywords: Stability region, Fractional dynamic, Unified system, State feedback controller.

1. INTRODUCTION

Fractional calculus is a mathematical topic with more than 300 years old history but the application to physics and engineering has been reported only in the recent years. It has been found that the behaviour of many physical systems can be properly described, using the fractional order system theory. Heat conduction, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, Visco-Elastic systems, quantum evolution of complex systems, quantitative finance and diffusion wave are among the known dynamical systems that were modelled using fractional order equations. In fact, real world processes generally or most likely are fractional order systems (Tavazoei and Haeri, 2009). Furthermore, fractional order controllers such as CRONE (Oustaloup et. al., 1996), TID (Lurie, 1994), fractional PID controller (Podlubny, 1999) and lead-lag compensator (Raynaud and Zerga Inoh, 2000) have been implemented to improve the performance and robustness of some closed loop control systems. An application of fractional algebra is to model the fractional order chaotic systems. This kind of modelling provides more accuracy, less complexity as well as the possibility to increase the stability region (Tavazoei and Haeri, 2009).

Chaos is a very interesting nonlinear phenomenon. High sensitivity to initial conditions is a main characteristic of chaotic systems. Accordingly, these systems are difficult for synchronization or control (Hosseinnia et. al., 2010). Due to complex behavior, and coupling, the control and stabilization task of chaotic nonlinear systems have been one of the major issues in control engineering area. In the past decade, a great efforts has been devoted towards the chaos control, including stabilization of unstable equilibrium points, and more generally, unstable periodic solutions. Particularly, in case of chaos suppression of known chaotic systems, some useful methods have been developed. These includes time delay feedback control (Pyragas and Tamasevieius, 1993), bang–bang control (Vincent, Yu, 1991), optimal control (Luce, Kernevez, 1991), intelligent control (Yeap, Ahmed, 1994), Adaptive control (Zeng, Singh, 1997) , etc.

The unified chaotic system is a chaotic system which depends on a parameter $\alpha \in [0,1]$. If $0 \le \alpha < 0.8$, the unified chaotic system reduces to the generalized Lorenz chaotic system; the unified chaotic system reduces to the Lü chaotic system when $\alpha = 0.8$. For $0.8 < \alpha \le 1$, the unified chaotic system reduces to the generalized Chen chaotic system.

Chen (Chen and Lu, 2002) considered that the parameter of the two unified chaotic systems is unknown and an adaptive controller was used to achieve synchronization based on Lyapunov stability theory. Chen (Chen et. al., 2004) investigated the stabilization and synchronization of the unified chaotic system via an impulsive control method. Lu (Lu et. al., 2004) used linear feedback and adaptive control to synchronize identical unified chaotic systems with only one controller. Ucar (Ucar et. al., 2006) used a nonlinear active controller to synchronize two coupled unified chaotic systems with three control inputs. Wang (Wang and Liu, 2007) proved that the unified chaotic system is equivalent to a passive system and asymptotically stabilized it at equilibrium points. Wang (Wang and Song, 2008) studied the synchronization problem of two identical unified chaotic systems using three different methods. They used a linear feedback controller, a nonlinear feedback method and an impulsive controller to synchronize the systems. In (Zribi et. al., 2009) based on the sliding mode theory synchronization of two identical unified chaotic is discussed. In this paper, a linear state feedback controller stabilizes fractional order unified chaotic system. The state feedback controller increases the stability region of fractional order chaotic system. An advantage of the proposed controller can be seen when it is used to stabilize a fractional order unified chaotic system. Meanwhile the application on the integer order system will be shown failed.

The paper is organized as follows. Section 2 includes the basic definition and preliminaries. State feedback controller is proposed to stabilize of fractional order unified chaotic systems in section 3. Results of numerical simulation are given in section 4, to illustrate the effectiveness of the proposed controller. The paper will be closed by a conclusion in section 5.

2. PRELIMINARY DEFINITIONS

2.1. Fractional Algebra

Among several definitions of fractional derivatives, the following Caputo-type definition is more popular with respect the rest (Caputo, 1967).

$$ {}_0D_t^q f(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)}\displaystyle\int_0^t \dfrac{f^m(\tau)}{(t-\tau)^{q+1-m}}\,d\tau, & m-1<q\le m \\ \dfrac{d^m}{dt^m} f(t) & q=m \end{cases} \tag{1} $$

where m is the first integer number larger than q.

Definition 1. (Tavazoei and Haeri, 2008) A saddle point of index 2 is a saddle point with one stable eigenvalue and two unstable ones.

Definition 2. (Tavazoei and Haeri, 2008) Assume that a 3-D fractional order chaotic system of $\dot{x} = f(x)$ displays a chaotic attractor. For every scroll existing in the chaotic attractor, this system has a saddle point of index 2 encircled by its respective scroll.

Theorem 1. (Tavazoei and Haeri, 2007) Assume that a 3-D chaotic system $\dot{x} = f(x)$ displays a chaotic attractor with n scrolls. Suppose Λ is a set of unstable eigenvalues of these n saddle points. A necessary condition for fractional system $D^q x = f(x)$ to exhibit an n-scroll chaotic attractor, similar to the chaotic attractor of system $\dot{x} = f(x)$, to keep the eigenvalues $\lambda \in \Lambda$ in the unstable region, satisfies:

$$ q > \frac{2}{\pi}\tan^{-1}\left(\frac{|\mathrm{Im}(\lambda)|}{\mathrm{Re}(\lambda)}\right), \qquad \forall \lambda \in \Lambda \tag{2} $$

Otherwise, at least one of these equilibriums becomes asymptotically stable and then attracts the nearby trajectories.

2.2. Unified Chaotic System

In (Lü et. al., 2002) considered a kind of chaotic system which describes a class of unified form which is as follows:

$$ \begin{cases} \dfrac{dx}{dt} = (25\alpha+10)(y-x) \\ \dfrac{dy}{dt} = (28-35\alpha)x - xz + (29\alpha-1)y \\ \dfrac{dz}{dt} = xy - \dfrac{8+\alpha}{3}z \end{cases} \tag{3} $$

where x, y, z are state variables and $\alpha \in [0,1]$ is parameter of system. Lü et. al. (Lü et. al., 2002) calls (3) as unified chaotic system because system (3) is chaotic for any $\alpha \in [0,1]$. When $0 \le \alpha < 0.8$, system (3) is called as the generalized Lorenz chaotic system. For $\alpha = 0.8$, it is called Lü chaotic system. Similarly it is called generalized Chen chaotic system when $0.8 < \alpha \le 1$. Besides, let us introduce the fractional version of equation (4). Standard derivatives of equation (3) are replaced by the following fractional derivatives:

$$ \begin{cases} \dfrac{d^q x}{dt^q} = (25\alpha+10)(y-x) \\ \dfrac{d^q y}{dt^q} = (28-35\alpha)x - xz + (29\alpha-1)y \\ \dfrac{d^q z}{dt^q} = xy - \dfrac{8+\alpha}{3}z \end{cases} \tag{4} $$

where q with $0 < q \le 1$ is the fractional order. Chaotic behaviour of fractional order unified system of Chen, Lü and Lorenz-Like for $q = 0.9, 0.95, 0.99$ are shown in (Matouk, 2009). From equation (4) the generalised scheme of fractional order unified chaotic system can be give as follows:

$$ \begin{cases} \dfrac{d^q x}{dt^q} = a(y-x) \\ \dfrac{d^q y}{dt^q} = bx - xz + cy \\ \dfrac{d^q z}{dt^q} = xy - dz \end{cases} \tag{5} $$

3. STATE FEEDBACK CONTROL

3.1. Design of the Controller for Fractional order Chen System

From (Matouk, 2009) the fractional order Chen system is given as follows:

$$ \begin{cases} \dfrac{d^q x}{dt^q} = a_1(y-x) \\ \dfrac{d^q y}{dt^q} = (c_1-a_1)x - xz + c_1 y \\ \dfrac{d^q z}{dt^q} = xy - b_1 z \end{cases} \tag{6} $$

To obtain the Chen chaotic behaviour, parameters in equation (6) is set to (Matouk, 2009):

$$a_1 = 40\ ,\ b_1 = 3\ ,\ c_1 = 28 \tag{7}$$

From equations (6) and (7) the equilibrium points of Chen system are given by:

$$\begin{aligned} &O_1 = (0,0,0) \\ &O_2 = (6.9282, 6.9282, 16) \\ &O_3 = (-6.9282, -6.9282, 16) \end{aligned} \tag{8}$$

From equation (6) Jacobian matrix of Chen system is achieved which is as follows:

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ c_1 - a_1 - z & c_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{9}$$

Accordingly, the corresponding eigenvalues of the equilibrium (8) are obtained as:

$$\begin{aligned} &O_1 \to \lambda_1 = -3, \lambda_2 = 20, \lambda_3 - 32 \\ &O_{2,3} \to \lambda_1 = -20.2304, \lambda_{2,3} = 2.6152 \pm 13.5268j \end{aligned} \tag{10}$$

From definition 1, $O_{2,3}$ are of saddle point of index 2. Therefore from theorem 1 fractional order Chen system becomes chaotic when:

$$q > \frac{2}{\pi} \tan^{-1}\left(\frac{\left|\mathrm{Im}(\lambda_{2,3})\right|}{\mathrm{Re}(\lambda_{2,3})} \right) = 0.8784 \tag{11}$$

Otherwise the system is asymptotically stable.

In order to stabilize the fractional order Chen system, an input signal is added into system dynamic by the following:

$$\begin{cases} \dfrac{d^q x}{dt^q} = a_1(y - x) \\ \dfrac{d^q y}{dt^q} = (c_1 - a_1)x - xz + c_1 y + u \\ \dfrac{d^q z}{dt^q} = xy - b_1 z \end{cases} \tag{12}$$

A linear state feedback controller is proposed to configure the input signal *u* as in the following form:

$$u = -(c_1 - a_1)x - k_1 y \tag{13}$$

where k_1 is a constant gain and $k_1 = 12.7$.

Theorem 2. The proposed state feedback controller in equation (13) increases the stability region of fractional order Chen system and stabilizes the system at their stable equilibrium points.

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Proof. Despite of the applied state feedback controller in fractional order Chen system the equilibrium points and the Jacobian matrix are obtained:

$$\begin{aligned} &O_1' = (0,0,0) \\ &O_2' = (6.7749, 6.7749, 15.3) \\ &O_3' = (-6.7749, -6.7749, 15.3) \end{aligned} \tag{14}$$

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ -z & c_1 - k_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{15}$$

Corresponding eigenvalues of the equilibrium points in equation (14) are:

$$\begin{aligned} &O_1' \to \lambda_1 = -3, \lambda_2 = 15.3, \lambda_3 = -40 \\ &O_{2,3}' \to \lambda_1 = -28.0829, \lambda_{2,3} = 0.1915 \pm 11.4331j \end{aligned} \tag{16}$$

Similarly, from the definition 1, $O_{2,3}'$ are of the saddle point of index 2. Hence the fractional order Chen system becomes chaotic when:

$$q > \frac{2}{\pi} \tan^{-1}\left(\frac{\left|\mathrm{Im}(\lambda_{2,3})\right|}{\mathrm{Re}(\lambda_{2,3})} \right) = 0.9893 \tag{17}$$

Otherwise system is asymptotically stable. This means that for $q < 0.9893$ the fractional order Chen system is asymptotically stable. ■

3.2. Design of the Controller for Fractional order Lü System

From (Matouk, 2009) the fractional order Lü system is given by:

$$\begin{cases} \dfrac{d^q x}{dt^q} = a_1(y - x) \\ \dfrac{d^q y}{dt^q} = -xz + c_1 y \\ \dfrac{d^q z}{dt^q} = xy - b_1 z \end{cases} \tag{18}$$

To obtain the Lü chaotic behaviour, parameters in equation (18) is set to (Matouk, 2009):

$$a_1 = 35\ ,\ b_1 = 3\ ,\ c_1 = 30 \tag{19}$$

From equations (18) and (19) the equilibrium points and Jacobian matrix of Lü system are given by:

$$\begin{aligned} &O_1 = (0,0,0) \\ &O_2 = (9.4868, 9.4868, 30) \\ &O_3 = (-9.4868, -9.4868, 30) \end{aligned} \tag{20}$$

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ -z & c_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{21}$$

Then the corresponding eigenvalues of the equilibrium points in equation (20) are:

$$\begin{aligned} &O_1 \to \lambda_1 = -3, \lambda_2 = 30, \lambda_3 = -35 \\ &O_{2,3} \to \lambda_1 = -19.3701, \lambda_{2,3} = 5.6851 \pm 17.1149\, j \end{aligned} \tag{22}$$

From the definition 1, $O_{2,3}$ are of the saddle point of index 2. Thus the fractional order Lü system becomes chaotic when:

$$q > \frac{2}{\pi} \tan^{-1}\left(\frac{|\mathrm{Im}(\lambda_{2,3})|}{\mathrm{Re}(\lambda_{2,3})} \right) = 0.7958 \tag{23}$$

Otherwise system is asymptotically stable.

Similar to the previous section, to stabilize the fractional order Lü system an input signal is added as a controller in the system dynamic which is as follows:

$$\begin{cases} \dfrac{d^q x}{dt^q} = a_1(y - x) \\ \dfrac{d^q y}{dt^q} = -xz + c_1 y + u \\ \dfrac{d^q z}{dt^q} = xy - b_1 z \end{cases} \tag{24}$$

The linear state feedback controller u in the following form stabilizes the chaotic dynamic:

$$u = -k_1 y \tag{25}$$

where k_1 is a constant gain of $k_1 = 16.5$.

Theorem 3. By the proposed state feedback controller in equation (25), the stability region of fractional order Lü system is increased and stabilizes the system at their stable equilibrium points.

Proof. Despite of state feedback controller in fractional order Lü system similar of equation (24) the equilibrium points and Jacobian matrix are achieved by:

$$\begin{aligned} &O'_1 = (0,0,0) \\ &O'_2 = (6.3639, 6.3639, 13.5) \\ &O'_3 = (-6.3639, -6.3639, 13.5) \end{aligned} \tag{26}$$

$$J = \begin{bmatrix} -a_1 & a_1 & 0 \\ -z & c_1 - k_1 & -x \\ y & x & -b_1 \end{bmatrix} \tag{27}$$

So the corresponding eigenvalues of the equilibrium points in equation (26) are:

$$\begin{aligned} &O'_1 \to \lambda_1 = -3, \lambda_2 = 13.5, \lambda_3 = -35 \\ &O'_{2,3} \to \lambda_1 = -24.863, \lambda_{2,3} = 0.1815 \pm 10.6767\, j \end{aligned} \tag{28}$$

Again from the definition 1, $O'_{2,3}$ are the saddle points of index 2 so from theorem 1, fractional order Lü system is chaotic when:

$$q > \frac{2}{\pi} \tan^{-1}\left(\frac{|\mathrm{Im}(\lambda_{2,3})|}{\mathrm{Re}(\lambda_{2,3})} \right) = 0.9892 \tag{29}$$

Otherwise system is asymptotically stable. This means that for $q < 0.9892$ the fractional order Lü system is asymptotically stable. ■

4. SIMULATION

A simulation approach has been carried out using SIMULINK™. Dormand-Prince solver is used to solve the system of differential equations during the simulation. Results of unified chaotic Chen and Lü systems are shown for $q = 0.96$, $q = 0.98$, $q = 1$. Initial conditions of the states are selected as $(10, 15, 25)$. Simulation results show that the simple state feedback controller stabilized the fractional order unified chaotic systems whilst the behaviour of the equivalent integer one still kept chaotic. Fig. 1 shows that the fractional order Chen system is stabilized for $q = 0.96$ and $q = 0.98$ with state feedback controller in equation (13). Fig. 2 shows the chaotic behaviour of integer order Chen system, despite of using the same state feedback controller in the system. Similar result is achieved in Fig. 3 when the fractional order Lü system is stabilized by the controller for $q = 0.96$ and $q = 0.98$. In the same way, Fig. 4 shows the chaotic behaviour of integer order of Lü system using the same state feedback controller.

5. CONCLUSION

Three chaotic Lorenz, Chen and Lü systems are shown unified by a same dynamic. Theses system will separately be excited when the relevant parameter α is accordingly adjusted. A simple linear state feedback controller is gained to stabilize the unified chaotic systems at their stable equilibrium points. The controller also increases the stability region with respect to their integer order counterpart. Simulation approach is given to verify the outcome. The approach signifies the performance as well as the reliability of the state feedback controller.

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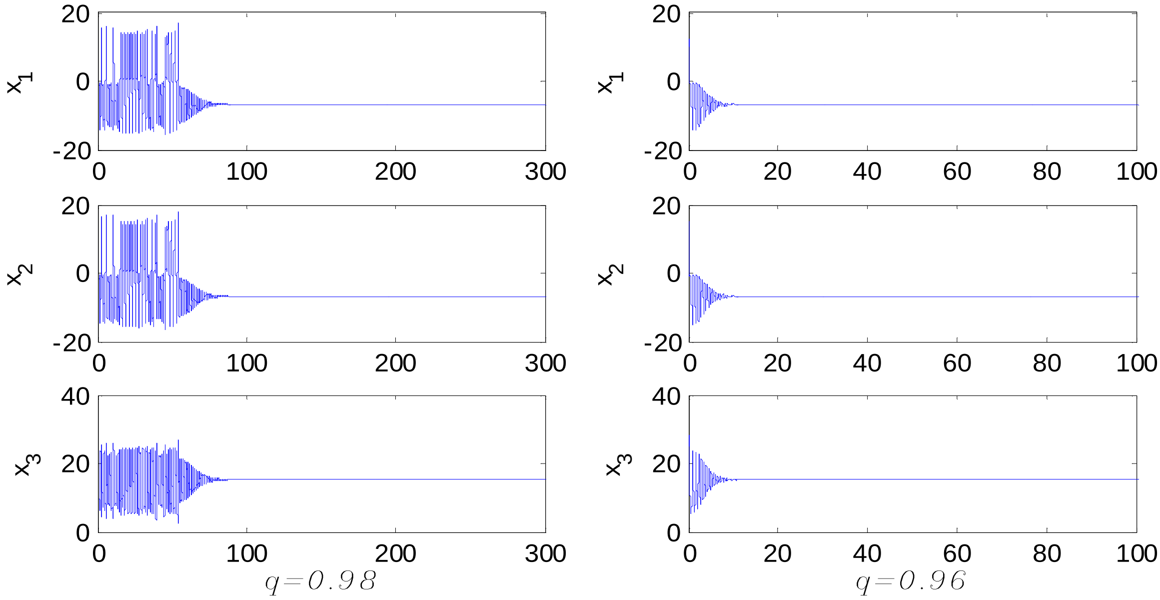

Fig. 1. Stabilize of fractional order Chen system via linear state feedback controller at their stable equilibrium points (O_2' and O_3') for $q = 0.96$ and $q = 0.98$

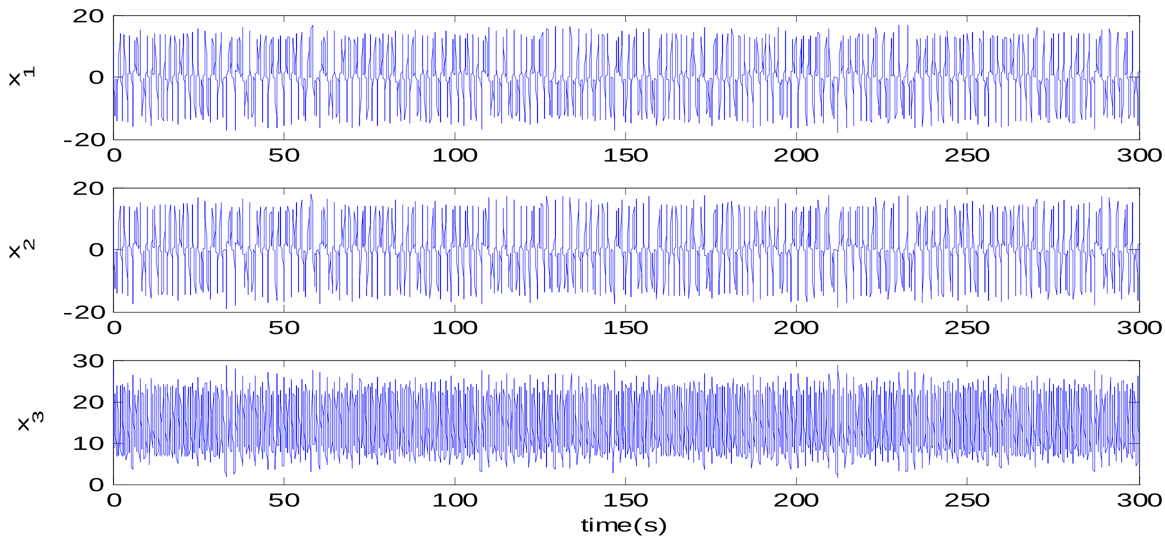

Fig. 2. Chaotic behavior of integer order of Chen system despite of state feedback controller

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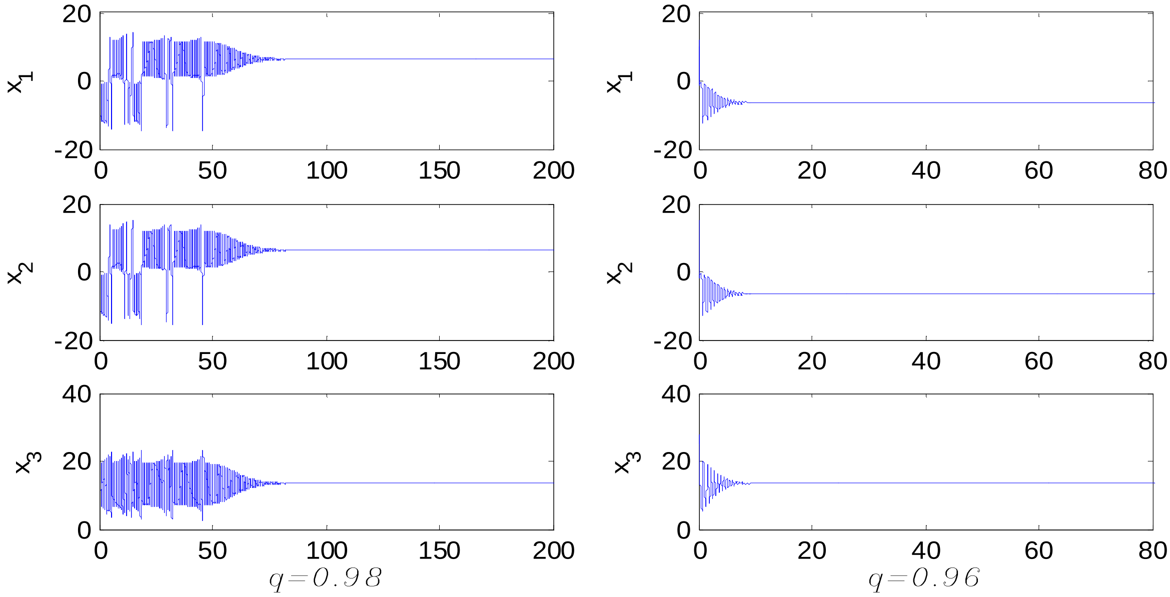

Fig. 3. Stabilize of fractional order Lü system via linear state feedback controller at their stable equilibrium points (O_2' and O_3') for $q = 0.96$ and $q = 0.98$

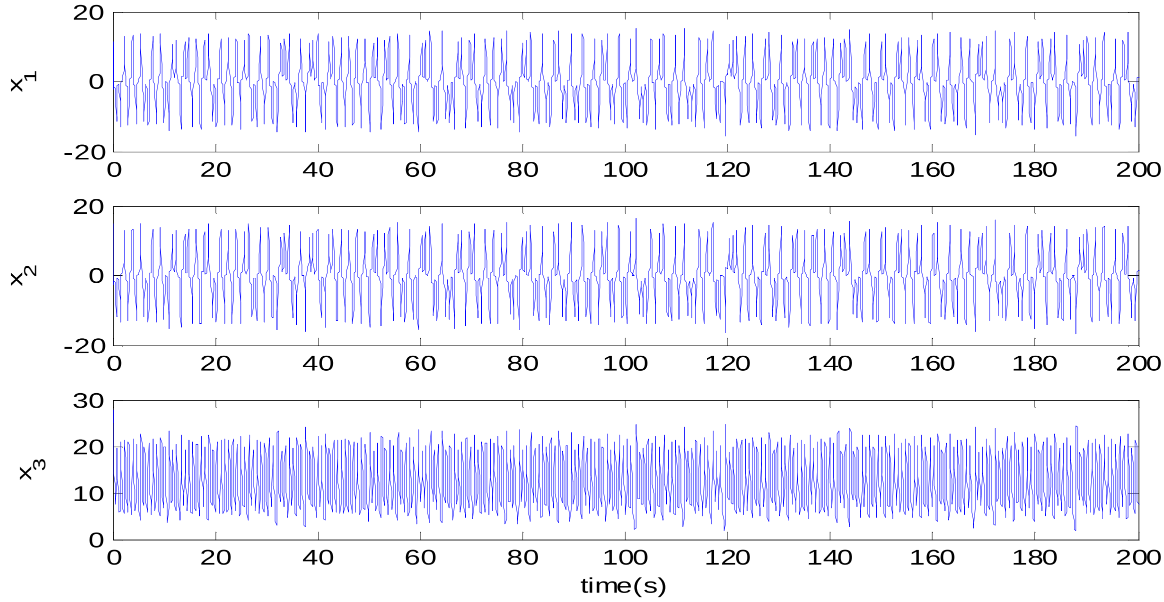

Fig. 4. Chaotic behavior of integer order of Lü system despite of state feedback controller

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