A Combined Discrete Fractional Sum Operator ⋆

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Abstract: We obtain existing results on delta and nabla discrete fractional sums by introducing a more general operator as a convex linear combination of the delta and nabla fractional sums. Main properties of the new operator are given.

Keywords: discrete fractional calculus, delta and nabla operators, convex linear combination.

1. INTRODUCTION

The main goal of this short note is to introduce a new more general fractional difference operator that unifies and extends the two discrete fractional operators used in the literature. Looking to the literature of discrete fractional difference operators, two approaches are found: one using the Δ operator (sometimes called the forward fractional difference operator), another using the ∇ operator (sometimes called the backward fractional difference operator). We introduce here a new operator, that we denote by ${}_\gamma\diamondsuit$ (cf. Definition 4). When $\gamma = 1$ the ${}_\gamma\diamondsuit$ operator reduces to the Δ one; when $\gamma = 0$ the ${}_\gamma\diamondsuit$ operator coincides with ∇. Basic properties of the new operator are given: cf. Theorems 7, 8, 11, and 15.

2. PRELIMINARIES

We give here only a very short introduction to the basic definitions in discrete fractional calculus. For more on the subject we refer the reader to Atici and Eloe (2007, 2009).

According with Bohner and Peterson (2001), we define the factorial function as

$$t^{(n)} = t(t-1)(t-2)\dots(t-n+1), \quad n \in \mathbb{N}.$$

According with the same authors Bohner and Peterson (2003), we define

$$t^{\overline{n}} = t(t+1)(t+2)\dots(t+n-1), \quad n \in \mathbb{N},$$

and $t^{\overline{0}} = 1$. Extending the two above definitions to an arbitrary real number α, we have

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)},$$

and

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}.$$

Throughout the text we shall use the notations $\sigma(s) = s+1$ and $\rho(s) = s-1$ of the time scale calculus Bohner and Peterson (2001).

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Definition 1. (Miller and Ross (1989)). The discrete delta fractional operator is defined by

$$(\Delta_a^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s),$$

where $\alpha > 0$. Here f is defined for $s = a \mod (1)$ and $\Delta_a^{-\alpha} f$ is defined for $t = (a+\alpha) \mod (1)$.

We note that $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$, where $\mathbb{N}_t = \{t, t+1, t+2, \dots\}$.

Analogously to Definition 1, one considers the discrete nabla fractional operator:

Definition 2. (Atıcı and Eloe (2009)). The discrete nabla fractional operator is defined by

$$(\nabla_a^{-\beta} f)(t) = \frac{1}{\Gamma(\beta)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\beta-1}} f(s),$$

where $\beta > 0$. Here f is defined for $s = a \mod (1)$ and $\nabla_a^{-\beta} f$ is defined for $t = a \mod (1)$.

The operator $\nabla_a^{-\beta}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$, where $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$.

The next result gives a relation between the delta fractional sum and the nabla fractional sum operators.

Lemma 3. (Atıcı and Eloe (2009)). Let $0 \le m-1 < \nu \le m$, where m denotes an integer. Let a be a positive integer, and $y(t)$ be defined on $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$. The following statement holds: $\Delta_a^{-\nu} y(t+\nu) = \nabla_a^{-\nu} y(t)$ for $t \in \mathbb{N}_a$.

3. MAIN RESULTS

Using a convex combination of the delta and nabla difference operators, we define the discrete diamond fractional sum.

Definition 4. The diamond-γ fractional operator of order (α, β) is given, when applied to a function f at point t, by

$$({}_\gamma\diamondsuit_a^{-\alpha,-\beta} f)(t) = \gamma \Delta_a^{-\alpha} f(t+\alpha) + (1-\gamma) \nabla_a^{-\beta} f(t),$$

where $\alpha > 0$, $\beta > 0$, and $\gamma \in [0,1]$. Here, both f and ${}_\gamma\diamondsuit_a^{-\alpha,-\beta} f$ are defined for $t = a \mod (1)$.

Remark 5. The operator ${}_{\gamma}\diamondsuit_a^{-\alpha,-\beta}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$, $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$.

Remark 6. The new diamond fractional operator of Definition 4 gives, as particular cases, the operator of Definition 1 and the operator of Definition 2,

$$\gamma = 1: \ {}_{1}\diamondsuit_a^{-\alpha,-\beta} f(t) = \Delta_a^{-\alpha} f(t+\alpha), \quad t \equiv a \mod (1),$$

and

$$\gamma = 0: \ {}_{0}\diamondsuit_a^{-\alpha,-\beta} f(t) = \nabla_a^{-\beta} f(t)\ , \quad t \equiv a \mod (1).$$

Our next theorems give important properties of the ${}_{\gamma}\diamondsuit_a^{-\alpha,-\beta}$ operator.

Theorem 7. Let f and g be real functions defined on $\mathbb{N}_a$. The following equality holds:

$$({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta})(f+g)(t) = {}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} f(t) + {}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} g(t).$$

Proof. By definition of diamond-γ fractional sum of order (α, β), we have

$$\begin{aligned}
&({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta})(f+g)(t)\\
&= (\gamma\Delta_a^{-\alpha}(f+g))(t+\alpha)\\
&\quad + (1-\gamma)(\nabla_a^{-\beta}(f+g))(t)\\
&= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+\alpha-\sigma(s))^{(\alpha-1)}(f(s)+g(s))\\
&\quad + \frac{1-\gamma}{\Gamma(\beta)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\beta-1}}(f(s)+g(s))\\
&= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+\alpha-\sigma(s))^{(\alpha-1)} f(s)\\
&\quad + \frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+\alpha-\sigma(s))^{(\alpha-1)} g(s)\\
&\quad + \frac{1-\gamma}{\Gamma(\beta)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\beta-1}} f(s)\\
&\quad + \frac{1-\gamma}{\Gamma(\beta)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\beta-1}} g(s)\\
&= \left[\frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+\alpha-\sigma(s))^{(\alpha-1)} f(s)\right.\\
&\quad \left. + \frac{1-\gamma}{\Gamma(\beta)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\beta-1}} f(s)\right]\\
&\quad + \left[\frac{\gamma}{\Gamma(\alpha)} \sum_{s=a}^{t} (t+\alpha-\sigma(s))^{(\alpha-1)} g(s)\right.\\
&\quad \left. + \frac{1-\gamma}{\Gamma(\beta)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\beta-1}} g(s)\right]\\
&={}_{\gamma}\ \diamondsuit_a^{-\alpha,-\beta} f(t) +{}_{\gamma}\ \diamondsuit_a^{-\alpha,-\beta} g(t).
\end{aligned}$$

Theorem 8. Let $f(t) = k$ be defined on $\mathbb{N}_a$, k a constant. The following equality holds:

$$\begin{aligned}
({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} k)(t) &= \gamma \frac{\Gamma(t-a+1+\alpha)}{\Gamma(\alpha+1)\Gamma(t-a+1)} k\\
&\quad + (1-\gamma)\frac{\Gamma(t-a+1+\beta)}{\Gamma(\beta+1)\Gamma(t-a+1)} k\,.
\end{aligned}$$

Proof. By definition of the diamond-γ fractional sum of order (α, β), we have

$$\begin{aligned}
&({}_{\gamma}\diamondsuit_a^{-\alpha,-\beta} k)(t) = \gamma(\Delta_a^{-\alpha} k)(t+\alpha) + (1-\gamma)(\nabla_a^{-\beta} k)(t)\\
&= \frac{\gamma}{\Gamma(\alpha)} \sum_{s=0}^{t} k(t+\alpha-\sigma(s))^{(\alpha-1)} + \frac{1-\gamma}{\Gamma(\beta)} \sum_{s=0}^{t} k(t-\rho(s))^{\overline{\beta-1}}\\
&= \gamma \frac{\Gamma(t-a+1+\alpha)}{\alpha\Gamma(\alpha)\Gamma(t-a+1)} k + (1-\gamma)\frac{\Gamma(t-a+1+\beta)}{\beta\Gamma(\beta)\Gamma(t-a+1)} k\\
&= \gamma \frac{\Gamma(t-a+1+\alpha)}{\Gamma(\alpha+1)\Gamma(t-a+1)} k + (1-\gamma)\frac{\Gamma(t-a+1+\beta)}{\Gamma(\beta+1)\Gamma(t-a+1)} k.
\end{aligned}$$

Remark 9. If $\gamma = 1$, then

$$\begin{aligned}
({}_{1}\diamondsuit_a^{-\alpha,-\beta} k)(t) &= \frac{\Gamma(t-a+1+\alpha)}{\Gamma(\alpha+1)\Gamma(t-a+1)} k\\
&= (\Delta_a^{-\alpha} k)(t+\alpha).
\end{aligned} \tag{1}$$

Note that when we take $a = 0$, (1) coincides with the result in Miller and Ross (1989).

Remark 10. If $\gamma = 0$, then

$$\begin{aligned}
({}_{0}\diamondsuit_a^{-\alpha,-\beta} k)(t) &= \frac{\Gamma(t-a+1+\beta)}{\Gamma(\beta+1)\Gamma(t-a+1)} k\\
&= (\nabla_a^{-\beta} k)(t).
\end{aligned}$$

Theorem 11. Let f be a real valued function and let α_1, α_2, β_1, $\beta_2 > 0$. Then,

$$\begin{aligned}
&{}_{\gamma}\diamondsuit_a^{-\alpha_1,-\beta_1} \left({}_{\gamma}\diamondsuit_a^{-\alpha_2,-\beta_2} f\right)(t)\\
&\qquad = \gamma \left({}_{\gamma}\diamondsuit_a^{-(\alpha_1+\alpha_2),-(\beta_1+\alpha_2)} f\right)(t)\\
&\qquad\quad + (1-\gamma)\left({}_{\gamma}\diamondsuit_a^{-(\alpha_1+\beta_2),-(\beta_1+\beta_2)} f\right)(t).
\end{aligned}$$

Proof. Direct calculations show the intended relation:

$$\begin{aligned}
&{}_{\gamma}\diamondsuit_a^{-\alpha_1,-\beta_1}({}_{\gamma}\diamondsuit_a^{-\alpha_2,-\beta_2} f)(t)\\
&= \gamma(\Delta_a^{-\alpha_1}({}_{\gamma}\diamondsuit_a^{-\alpha_2,-\beta_2} f))(t+\alpha_1)\\
&\quad + (1-\gamma)(\nabla_a^{-\beta_1}({}_{\gamma}\diamondsuit_a^{-\alpha_2,-\beta_2} f))(t)\\
&= \gamma^2 \Delta_a^{-\alpha_1}\left(\Delta_a^{-\alpha_2} f\right)(t+\alpha_1+\alpha_2)\\
&\quad + \gamma(1-\gamma)\Delta_a^{-\alpha_1}\left(\nabla_a^{-\beta_2} f\right)(t+\alpha_1)\\
&\quad + (1-\gamma)\gamma\nabla_a^{-\beta_1}\left(\Delta_a^{-\alpha_2} f\right)(t+\alpha_2)\\
&\quad + (1-\gamma)^2\nabla_a^{-\beta_1}\left(\nabla_a^{-\beta_2} f\right)(t)\\
&= \gamma^2 \Delta_a^{-(\alpha_1+\alpha_2)} f(t+\alpha_1+\alpha_2)\\
&\quad + \gamma(1-\gamma)\Delta_a^{-\alpha_1}\left(\Delta_a^{-\beta_2} f\right)(t+\alpha_1+\beta_2)\\
&\quad + (1-\gamma)\gamma\nabla_a^{-\beta_1}\left(\nabla_a^{-\alpha_2} f\right)(t) + (1-\gamma)^2\nabla_a^{-(\beta_1+\beta_2)} f(t)\\
&= \gamma^2 \Delta_a^{-(\alpha_1+\alpha_2)} f(t+\alpha_1+\alpha_2)\\
&\quad + \gamma(1-\gamma)\Delta_a^{-(\alpha_1+\beta_2)} f(t+\alpha_1+\beta_2)\\
&\quad + (1-\gamma)\gamma\nabla_a^{-(\beta_1+\alpha_2)} f(t) + (1-\gamma)^2\nabla_a^{-(\beta_1+\beta_2)} f(t)\\
&= \gamma\left[\gamma\Delta_a^{-(\alpha_1+\alpha_2)} f(t+\alpha_1+\alpha_2)\right.\\
&\quad \left. + (1-\gamma)\nabla_a^{-(\beta_1+\alpha_2)} f(t)\right]\\
&\quad + (1-\gamma)\left[\gamma\Delta_a^{-(\alpha_1+\beta_2)} f(t+\alpha_1+\beta_2)\right.\\
&\quad \left. + (1-\gamma)\nabla_a^{-(\beta_1+\beta_2)} f(t)\right].
\end{aligned}$$

Remark 12. If $\gamma = 0$, then

$$\left({}_{\gamma}\diamondsuit_a^{-\alpha_1,-\beta_1}({}_{\gamma}\diamondsuit_a^{-\alpha_2,-\beta_2} f)\right)(t) = \left(\nabla_a^{-(\beta_1+\beta_2)} f\right)(t).$$

Remark 13. If $\gamma = 1$, then

$$\left({}_{\gamma}\diamondsuit_a^{-\alpha_1,-\beta_1}({}_{\gamma}\diamondsuit_a^{-\alpha_2,-\beta_2} f)\right)(t) = \left(\Delta_a^{-(\alpha_1+\alpha_2)} f\right)(t+\alpha_1+\alpha_2).$$

Remark 14. If $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, then

$$\left({}_\gamma\Diamond_a^{-\alpha_1,-\beta_1}\left({}_\gamma\Diamond_a^{-\alpha_2,-\beta_2} f\right)\right)(t) = \left({}_\gamma\Diamond_a^{-\alpha,-\beta} f\right)(t).$$

Theorem 15. Let f and g be real valued functions, $0 < \alpha,\ \beta < 1$. Then, for all t such that $t = a \mod (1)$, the following equality holds:

$$\begin{aligned}
&{}_\gamma\Diamond_a^{-\alpha,-\beta}(fg)(t)\\
&\quad = \gamma\sum_{k=0}^{\infty}\binom{-\alpha}{k}\left[\nabla^k g(t)\right]\cdot\left[\Delta_a^{-(\alpha+k)} f(t+\alpha+k)\right]\\
&+ (1-\gamma)\sum_{k=0}^{\infty}\binom{-\beta}{k}\left[\nabla^k g(t)\right]\left[\Delta_a^{-(\beta+k)} f(t+\beta+k)\right],
\end{aligned}$$

where $t \equiv a \mod 1$ and

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$$

for $0 < u, v < 1$ and k a nonnegative integer.

Proof. By the definition of the diamond fractional sum,

$$\begin{aligned}
&{}_\gamma\Diamond_a^{-\alpha,-\beta}(fg)(t) = \gamma\Delta_a^{-\alpha}(fg)(t+\alpha) + (1-\gamma)\nabla_a^{-\beta}(fg)(t)\\
&= \frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)} f(s)g(s)\\
&+ \frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}} f(s)g(s).
\end{aligned}$$

By Taylor expansion of $g(s)$ Anderson (2002), we have

$$g(s) = \sum_{k=0}^{\infty}\frac{(s-t)^{\overline{k}}}{k!}\nabla^k g(t) = \sum_{k=0}^{\infty}(-1)^k\frac{(t-s)^{(k)}}{k!}\nabla^k g(t).$$

Substituting the Taylor series of $g(s)$ at t in the sums,

$$\begin{aligned}
{}_\gamma\Diamond_a^{-\alpha,-\beta}(fg)(t) &= \frac{\gamma}{\Gamma(\alpha)}\sum_{s=a}^{t}(t+\alpha-\sigma(s))^{(\alpha-1)} f(s)\\
&\cdot\left[\sum_{k=0}^{\infty}(-1)^k(t-s)^{(k)}\frac{\nabla^k g(t)}{k!}\right]\\
&+\frac{1-\gamma}{\Gamma(\beta)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\beta-1}} f(s)\\
&\cdot\left[\sum_{k=0}^{\infty}(-1)^k(t-s)^{(k)}\frac{\nabla^k g(t)}{k!}\right].
\end{aligned}$$

Since

$$\begin{aligned}
(t+\alpha-\sigma(s))^{(\alpha-1)}(t-s)^{(k)} &= (t+\alpha-\sigma(s))^{(\alpha+k+1)},\\
(t-\rho(s))^{\overline{\beta-1}}(t-s)^{(k)} &= (t+\beta-\sigma(s))^{(\beta+k+1)},
\end{aligned}$$

and $\displaystyle\sum_{s=t-k+1}^{t}(t-s)^{(k)} = 0$, we have

$$\begin{aligned}
&{}_\gamma\Diamond_a^{-\alpha,-\beta}(fg)(t)\\
&= \frac{\gamma}{\Gamma(\alpha)}\sum_{k=0}^{\infty}(-1)^k\frac{\nabla^k g(t)}{k!}\sum_{s=a}^{t-k}(t+\alpha-\sigma(s))^{(\alpha+k-1)} f(s)\\
&+ \frac{1-\gamma}{\Gamma(\beta)}\sum_{k=0}^{\infty}(-1)^k\frac{\nabla^k g(t)}{k!}\sum_{s=a}^{t-k}(t+\beta-\sigma(s))^{(\beta+k-1)} f(s).
\end{aligned}$$

Because

$$(-1)^k = \frac{\Gamma(-\alpha+1)\Gamma(\alpha)}{\Gamma(-\alpha+k+1)\Gamma(k+\alpha)},$$

$$(-1)^k = \frac{\Gamma(-\beta+1)\Gamma(\beta)}{\Gamma(-\beta+k+1)\Gamma(k+\beta)},$$

and $k! = \Gamma(k+1)$, the above expression becomes

$$\begin{aligned}
&{}_\gamma\Diamond_a^{-\alpha,-\beta}(fg)(t)\\
&= \frac{\gamma}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\nabla^k g(t)\binom{-\alpha}{k}\\
&\cdot\left[\frac{1}{\Gamma(k+\alpha)}\sum_{s=a}^{t-k}(t+\alpha-\sigma(s))^{(\alpha+k-1)} f(s)\right]\\
&+ \frac{1-\gamma}{\Gamma(\beta)}\sum_{k=0}^{\infty}\nabla^k g(t)\binom{-\beta}{k}\\
&\cdot\left[\frac{1}{\Gamma(k+\beta)}\sum_{s=a}^{t-k}(t+\beta-\sigma(s))^{(\beta+k-1)} f(s)\right]\\
&= \gamma\sum_{k=0}^{\infty}\binom{-\alpha}{k}\nabla^k g(t)\Delta_a^{-(\alpha+k)} f(t+\alpha+k)\\
&+ (1-\gamma)\sum_{k=0}^{\infty}\binom{-\beta}{k}\nabla^k g(t)\Delta_a^{-(\beta+k)} f(t+\beta+k).
\end{aligned}$$

Remark 16. If $\gamma = 0$, then

$$\begin{aligned}
{}_0\Diamond_a^{-\alpha,-\beta}(fg)(t) &= \nabla_a^{-\beta}(fg)(t)\\
&= \sum_{k=0}^{\infty}\binom{-\beta}{k}\left[\nabla^k g(t)\right]\left[\Delta_a^{-(\beta+k)} f(t+\beta+k)\right].
\end{aligned}$$

Remark 17. If $\gamma = 1$, then

$$\begin{aligned}
{}_1\Diamond_a^{-\alpha,-\beta}(fg)(t) &= \Delta_a^{-\alpha}(fg)(t+\alpha)\\
&= \sum_{k=0}^{\infty}\binom{-\alpha}{k}\left[\nabla^k g(t)\right]\left[\Delta_a^{-(\alpha+k)} f(t+\alpha+k)\right]. \quad (2)
\end{aligned}$$

Additionally, let $a = 0$. Then, recalling Lemma 3, we note that (2) coincides with the Leibniz formulas in Atici and Senguel (2010).

4. CONCLUSION

The discrete fractional calculus is a subject under strong current research: see, e.g., Baleanu and Jarad (2006); Bastos et al. (2010); Cresson et al. (2009).

Two versions of the discrete fractional calculus, the delta and the nabla, are now standard in the fractional theory. Here, motivated by the diamond-alpha dynamic derivative on time scales Sidi Ammi et al. (2008); Malinowska and Torres (2009); Mozyrska and Torres (2009), we introduce a combined diamond-gamma fractional sum of order alpha, beta as a linear combination of the delta and nabla fractional operators of order alpha and beta, respectively. The new operator interpolates between the delta and nabla cases, reducing to the standard fractional delta operator when $\gamma = 1$ and to the fractional nabla sum for $\gamma = 0$.

It is our intention to generalize the new discrete diamond fractional operator we introduce here to arbitrary time scales. Another line of research, to be addressed elsewhere, consists to investigate the usefulness of modeling with fractional diamond equations and corresponding fractional variational problems.

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