The power functions and their Fourier transforms

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Abstract: The importance of the power functions in fractional calculus motivates this paper. They are presented and their Fourier transforms computed. The difference between the causal and the acausal (two-sided) powers is enhanced when comparing the corresponding transforms. A table is presented with the computed transforms.

Keywords: fractional derivative; power function; Fourier transform.

1. INTRODUCTION

The exponential function is the fundamental tool for dealing with integer order systems, but when entering into the context of the fractional systems its role is assumed by the power function. While the exponential function is connected with the short memory concept, the power function is characterized by a long memory, or as it is frequently called, a long range dependence. Although the usual tool for studying linear systems is the Laplace transform (LT), there are situations where we cannot use the LT, or where the information it gives is not required. In these situations the Fourier transform (FT) can be a valid alternative. However, the FT may lead in some situations to conflicting or inconsistent results, mainly when working in the context of Distributions – generalised functions (Ferreira, 1997, Hoskins and Pinto, 1994; Silva, 1989). In these cases, it is sometimes difficult to find the FT for some power functions, and unless care is taken, it is also difficult to find the correct result. In this paper we shall be working in the context of the axiomatic theory of distributions (Ferreira, 1997) due to its simplicity and engineering "flavour".

In this paper, we present a complete and correct table of expressions for the FT of integer and fractional order power functions. In the section 2, we present the most general formulations of the fractional derivatives. These operators, when looked from the point of view of the linear systems, have impulse responses that are power functions: either causal or acausal. In sections 3 we study the causal case and in section 4 the acausal case. Finally, we will present some conclusions. A table with all the transforms is in the appendix.

2. THE FRACTIONAL DERIVATIVES

2.1 Definitions

In this paper we deal with the multi-valued expression z^α. As is well known, in order to define a power function of a complex variable we have to fix a branch cut line and choose a branch (Riemann surface). It is a common procedure to choose the negative real half-axis as branch cut line. In what follows we will assume that we can adopt the principal branch and assume that the obtained function is continuous above the branch cut line. Using this convention, we will write $(-1)^\alpha = e^{j\alpha\pi}$. In addition, in all the following we shall be working with functions defined in the whole real number line.

The general formulation of the incremental ratio valid for any order, real or complex, leads to the definition of the fractional derivative as:

$$D_\theta^\alpha f(z) = e^{-j\theta\alpha} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{|h|^\alpha}, \tag{1}$$

where $h = |h|e^{j\theta}$ is a complex number, with $\theta \in (-\pi,\pi]$ and $\theta \neq \pi/2$. The above definition of the fractional derivative as a general incremental ratio generalizes the classical Grünwald-Letnikov fractional derivative. Here, we will assume that h is real and that $\theta = 0$, or $\theta = \pi$. If $\theta = 0$, only the present and past values are being used, while, if $\theta = \pi$, only the present and future values are used. This means that if we look at (1) as a linear system, the first case is causal, while the second is anti-causal (Ortigueira, 2006a,b,2008) Usually, if $\theta = 0$, we call (1) the forward Grünwald-Letnikov derivative, as shown below:

$$D_f^\alpha f(z) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha} \tag{2}$$

We know that considering this derivative as a system, we can show (Ortigueira, 2006a) that the transfer function is s^α, for $\mathrm{Re}(s) > 0$ and the frequency response is given by Ortigueira and Trujillo, (2009) as:

$$(j\omega)^\alpha = |\omega|^\alpha \begin{cases} e^{j\alpha\pi/2} & \text{if } \omega>0 \\ e^{-j\alpha\pi/2} & \text{if } \omega<0 \end{cases} \tag{3}$$

The corresponding impulse response will be presented later.

Another derivative formulation leads to the centred derivatives (Ortigueira, 2006b, 2008). Assuming again that $\alpha > -1$, we have

$$D^{\alpha}_{c_1} f(t) = \lim_{h\to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha/2-k+1)\,\Gamma(\alpha/2+k+1)} f(t-kh) \tag{4}$$

that is, (4) represents the type 1 fractional central derivative. The type 2 fractional central derivative is given by

$$D^{\alpha}_{c_2} f(t) = \lim_{h\to 0} \frac{\Gamma(\alpha+1)}{h^{\alpha}} \sum_{-\infty}^{+\infty} \frac{(-1)^k f(t-kh+h/2)}{\Gamma[(\alpha+1)/2-k+1]\,\Gamma[(\alpha-1)/2+k+1]} \tag{5}$$

where $\alpha \neq 0$.

The linear systems defined by these derivatives do not have transfer functions, but they have frequency responses given by

$$H_{D1}(\omega) = |\omega|^{\alpha} \tag{6}$$

and

$$H_{D2}(\omega) = -j\,|\omega|^{\alpha}\mathrm{sgn}(\omega) \tag{7}$$

respectively (Ortigueira, 2006b, 2008).

2.2 The impulse responses

The impulse responses corresponding to the above introduced frequency responses are given respectively by:

$$h(t) = \frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)} \tag{8}$$

for the causal Grünwald-Letnikov derivative (here, u(t) is the Heaviside unit step), and

$$h_{D1}(t) = \frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} |t|^{-\alpha-1} \tag{9}$$

and

$$h_{D2}(t) = -\frac{\mathrm{sgn}(t)}{2\Gamma(-\alpha)\sin(\alpha\pi/2)} |t|^{-\alpha-1} \tag{10}$$

for the centred acausal derivatives. As shown in (Ortigueira and Trujillo, 2009) we have

$$FT\left[\frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)}\right] = |\omega|^{\alpha} e^{j\alpha\pi/2\,\mathrm{sgn}(\omega)} \tag{11}$$

where α is not a positive integer. We will write merely

$$FT\left[\frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)}\right] = (j\omega)^{\alpha} \tag{12}$$

With this, we are dealing with a multi-valued expression z^{α}. As is the common procedure, we choose the negative real half-axis as the branch cut line.

In what follows we will assume that we adopt the principal branch and assume that the obtained function is continuous above the branch cut line. This justifies the above expression and allows us to write $(-1)^{\alpha} = e^{j\alpha\pi}$. To obtain the expressions (11) and (12) we can compute the Laplace transform of (4) for $\alpha < 0$ as shown by Henrici (1991). On making the replacement, $s \to j\omega$ we migrate from the Laplace transform to the Fourier transform. We define the Fourier transform (FT) by

$$F(\omega) = FT[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \tag{13}$$

After we use the generalised integer order derivative as we will explain in the next section. With this we can conclude that (6) is valid for any real excepting the positive integers. In this case, the inverse Fourier transform of $(j\omega)^n$ is the n^{th} order derivative of the delta Dirac function. We can use (12) to obtain the Fourier transforms of (9) and (10). This can be done in two ways:

- By correlation of two powers as in (12) but with $\alpha/2$.
- By adding (12) and its time reversed version.

We will follow the second procedure. We have

$$\frac{|t|^{-\alpha-1}}{\Gamma(-\alpha)} = \frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)} + \frac{(-t)^{-\alpha-1}u(-t)}{\Gamma(-\alpha)} \tag{14}$$

that with (11) leads to

$$FT\left[\frac{|t|^{-\alpha-1}}{2\cos(\alpha\pi/2)\Gamma(-\alpha)}\right] = |\,\omega\,|^{\alpha} \tag{15}$$

Similarly, with

$$\frac{|t|^{-\alpha-1}}{\Gamma(-\alpha)} = \frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)} - \frac{(-t)^{-\alpha-1}u(-t)}{\Gamma(-\alpha)} \tag{16}$$

we obtain

$$FT\left[\frac{|t|^{-\alpha-1}\mathrm{sgn}(t)}{2\Gamma(-\alpha)\sin(\alpha\pi/2)}\right] = j|\,\omega\,|^{\alpha}\,\mathrm{sgn}(\omega) \tag{17}$$

In both (15) and (17) α is non-positive integer real. We can rewrite them as

$$FT\left[|t|^{-\alpha-1}\right] = 2\Gamma(-\alpha)\cos(\alpha\pi/2)|\,\omega\,|^{\alpha} \tag{18}$$

and

$$FT\left[|t|^{-\alpha-1}\mathrm{sgn}(t)\right] = 2j\Gamma(-\alpha)\sin(\alpha\pi/2)|\,\omega\,|^{\alpha}\,\mathrm{sgn}(\omega) \tag{19}$$

But the reflection formula of the gamma function allows us to obtain:

$$\frac{1}{2\Gamma(-\alpha)\cos(\alpha\pi/2)} = -\frac{\Gamma(\alpha+1)\sin(\alpha\pi/2)}{\pi} \tag{20}$$

and

$$\frac{1}{2\Gamma(-\alpha)\sin(\alpha\pi/2)} = -\frac{\Gamma(\alpha+1)\cos(\alpha\pi/2)}{\pi} \tag{21}$$

Letting n be a positive integer, leads to

$$FT\left[|t|^{-2n}\right] = \frac{(-1)^n\pi}{(2n-1)!}|\omega|^{2n-1} \quad (22)$$

and to

$$FT\left[|t|^{-2n-1}sgn(t)\right] = -j\frac{(-1)^n\pi}{(2n)!}|\omega|^{2n}sgn(\omega) \quad (23)$$

As $|t|^{-2n} = t^{-2n}$, $|t|^{-2n-1}sgn(t) = t^{-2n-1}$, we can write:

$$FT\left[t^{-2n}\right] = \frac{(-1)^n\pi}{(2n-1)!}|\omega|^{2n-1} \quad (24)$$

and

$$FT\left[t^{-2n-1}\right] = j\frac{(-1)^n\pi}{(2n)!}|\omega|^{2n}sgn(\omega) \quad (25)$$

2.3 The generalised functions and their Fourier transforms

We choose to follow the formalism presented in the Axiomatic Theory of Generalised Functions (Silva, 1989, Ferreira, 1997). This approach defines a generalised function or distribution as an integer order derivative of a continuous function. Let g(t) be a continuous function defined in R and f(t) be a function such that

$$f(t) = D^N g(t) \quad (26)$$

We say that f(t) is a distribution. For example,

$$\delta(t) = D^2[tu(t)] = 1/2D^2|t| \quad (27)$$

will lead to:

$$\delta(t) = Du(t) = 1/2D[sgn(t)] \quad (28)$$

where sgn(t) is the signum function. It can be proven that, if f(t) is a tempered distribution([1]), then its FT exists and it is also a tempered GF. In the following we shall be working with tempered distributions.

3. THE FOURIER TRANSFORMS OF ACAUSAL POWER FUNCTIONS

3.1 Positive integer order powers

We start by considering the acausal case, due to the symmetry that allows us to test the coherence of the results.

As we wrote above

$$\delta(t) = ½ D[sgn(t)] \quad (29)$$

[1] a generalised function φ is said to be tempered or of polynomial type if and only if $\exists\, \alpha \in R$ such that $\frac{\varphi(t)}{t^\alpha}$ is bounded when $t\to\infty$.

We prefer this approach because it maintains the even/odd relationship between a function and its derivative. Using the well known properties of the Fourier transform and those of the $\delta(t)$, we have

$$FT[\delta(t)] = 1 \quad (30)$$

and, if $S(\omega)$ is the FT of the signum function, we have

$$j\omega\, S(\omega) = 1$$

This equation has infinite solutions with the general format

$$S(\omega) = \frac{2}{j\omega} + C\,\delta(\omega) \quad (31)$$

As sgn(t) is odd, $S(\omega)$ must be odd. Then C=0 and

$$S(\omega) = \frac{2}{j\omega} \quad (32)$$

The product t.sgn(t) is well defined, because t is an ordinary continuous function. Then

$$FT[t\, sgn(t)] = j\frac{d}{d\omega}S(\omega) = -\frac{2}{\omega^2} \quad (33)$$

Repeating the process, we have

$$FT[t^2 sgn(t)] = 2j\frac{2}{\omega^3}\,;\ FT[t^3 sgn(t)] = -2\frac{6}{\omega^4}\ \ldots$$

If n is an odd integer, $n \to 2n+1$ (n=1,2, ...), $t^{2n+1}sgn(t) = |t|^{2n+1}$ and we have:

$$FT[|t|^{2n+1}] = -2\frac{(2n+1)!}{\omega^{2n+2}} \quad (34)$$

If n is an even integer, $n \to 2n$ (n=1,2, ...), we have:

$$FT[t^{2n}sgn(t)] = 2j\frac{(2n)!}{\omega^{2n+1}} \quad (35)$$

These two expressions state coherent results, since, if we derive one from one set, we obtain another from the other.

As

$$FT[1] = 2\pi\delta(\omega) \quad (36)$$

and

$$FT[t^k f(t)] = j^k F^{(k)}(\omega) \quad (37)$$

we obtain

$$FT[t^n] = 2\pi j^n \delta^{(n)}(\omega) \quad (38)$$

In particular

$$FT[t^{2n}] = FT[|t|^{2n}] = 2\pi(-1)^n \delta^{(2n)}(\omega) \quad (39)$$

By taking the derivative of the left hand side, we obtain:

$$FT[(2n)|t|^{2n-1}sgn(t)] = 2\pi(-1)^n j\omega.\delta^{(2n)}(\omega) \quad (40)$$

But (Ferreira, 1997)

$$\omega^k.\delta^{(m)}(\omega) = \begin{cases} 0 & \text{if } k>m \\ (-1)^k\frac{m!}{(m.k)!}\delta^{(m-k)}(\omega) & \text{if } k\le m \end{cases}$$

With k=1 and m=2n, we obtain

$$\omega.\delta^{(2n)}(\omega) = - \delta^{(2n-1)}(\omega)$$

and

$$FT[\, |t|^{2n+1}\mathrm{sgn}(t)\,] = 2\pi(-1)^{n+1}\, j\delta^{(2n+1)}(\omega) \qquad (41)$$

3.2 Odd negative integer orders

We first are going obtain the FT of odd integer negative powers. To see the problem we have to face, we are going to compute the FT of $|t|^{-1}$. We have

$$FT[t\frac{1}{|t|}] = FT[\mathrm{sgn}(t)] = \frac{2}{j\omega}$$

But

$$FT[t\frac{1}{|t|}] = j\frac{d}{d\omega}FT[\frac{1}{|t|}]$$

We have

$$j\frac{d}{d\omega}FT[\frac{1}{|t|}] = \frac{2}{j\omega}$$

leading to

$$FT[\frac{1}{|t|}] = -2\log|\,\omega\,| + C$$

where C is any real constant. We are going to consider the situation with all the generality. We can compute the derivative of both sides of the relation (19) relative to α. We obtain

$$FT\left[-|t|^{-\alpha-1}\log|t|\mathrm{sgn}(t)\right] = 2j\Gamma(-\alpha)\sin(\alpha\pi/2)|\omega|^{\alpha}\log|\omega|\,\mathrm{sgn}(\omega)+$$

$$+\, 2[-\Gamma(-\alpha)\sin(\alpha\pi/2) + \Gamma(-\alpha)\cos(\alpha\pi/2)\pi/2]\,|\omega|^{\alpha}\mathrm{sgn}(\omega) \qquad (42)$$

with α=-1, we obtain

$$FT\left[\log|t|\mathrm{sgn}(t)\right] = -2j\omega^{-1}[\log|\omega| + \gamma] \qquad (43)$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant. The finite part of the derivative of the first member in (43) gives us:

$$FT\left[\frac{1}{|t|}\right] = 2[\log|\omega| + \gamma] \qquad (44)$$

This equation is in agreement with the result obtained above. Continuing the derivative computation, we obtain successively:

$$FT\left[\frac{1}{|t|^2}\mathrm{sgn}(t)\right] = -\,2j\omega[\log|\omega| + \gamma],$$

$$FT\left[\frac{2}{|t|^3}\right] = 2(j\omega)^2[\log|\omega| + \gamma],$$

$$FT\left[\frac{6}{|t|^4}\mathrm{sgn}(t)\right] = -\,2(j\omega)^3[\log|\omega| + \gamma],$$

$$FT\left[\frac{24}{|t|^5}\right] = 2(j\omega)^4[\log|\omega| + \gamma],$$

which allows us to write:

$$FT\left[|t|^{-2n-1}\right] = \frac{2(-1)^n\omega^{2n}}{(2n)!}[\log|\omega| + \gamma] \qquad (45)$$

and

$$FT\left[|t|^{-2n}\mathrm{sgn}(t)\right] = \frac{2j(-1)^{n-1}\omega^{2n-1}}{(2n)!}[\log|\omega| + \gamma] \qquad (46)$$

To finish this problem, we need to compute the FT of $|\,t\,|^{-2n}$ and of $|\,t\,|^{-2n-1}\mathrm{sgn}(t)$. To do this, we remark that

$$|\,t\,|^{-2n} = t^{-2n}$$

and

$$|\,t\,|^{-2n-1}\mathrm{sgn}(t) = t^{-2n-1}$$

The corresponding FT are in (24) and (25) and the duality property of the FT. We conclude that the "abnormal" transform appears only in the situations stated in (45) and (46).

4. THE FOURIER TRANSFORMS OF CAUSAL POWER FUNCTIONS

4.1 Positive powers

Consider again the Fourier transform of the causal differintegrator given by (11). That expression is valid for any non-negative integer real, $\alpha \in R\text{-}Z^-$. In fact, for positive integer orders we obtain the repeated integration, while for negative integers we obtain the normal derivatives that have integer order derivatives of δ(t) as impulse responses. So, we intend to obtain the Fourier transform of $t^n u(t)$, for $n \in Z$. For positive n, we do not have great difficulties, because it is enough to start from the FT of the Heaviside unit step, multiply it by t^n and use a well known property of the FT.

We start from the relation

$$\delta(t) = D[u(t)] \qquad (47)$$

and use the fact that the equation

$$j\omega U(\omega)=1$$

has infinite solutions given by

$$U(\omega) = \frac{1}{j\omega} + A\delta(\omega)$$

where A must be non null since u(t) is not even neither odd; so, its FT must be complex. To be coherent with equation (31) above, A must be equal to π. To see why, we must remark that

$$u(t) = 2\mathrm{sgn}(t) - 1$$

and the FT of f(t) =1 is 2πδ(ω). Thus

$$FT[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega) \qquad (48)$$

Now, it is a simple task to write

$$FT[t^n u(t)] = (-j)^{n-1}\frac{1}{\omega^{n+1}} + \pi(-j)^n\delta^{(n)}(\omega) \quad (49)$$

if n is a positive integer. If n is negative, we must find another way.

To obtain the corresponding result for negative powers, we return back to equation (46). As sgn(t) = 2u(t) -1, we have:

$$|t|^{-2n}\mathrm{sgn}(t) = 2\, t^{-2n}u(t) - |t|^{-2n}$$

and

$$2\, t^{-2n}u(t) = |t|^{-2n} + |t|^{-2n}\mathrm{sgn}(t) = t^{-2n} + t^{-2n}\mathrm{sgn}(t)$$

Thus

$$FT[t^{-2n}u(t)] = \frac{(-1)^n\pi}{2(2n-1)!}|\,\omega\,|^{2n-1} + \frac{j(-1)^{n-1}\omega^{2n-1}}{(2n)!}[\log|\omega| + \gamma] \quad (50)$$

Similarly,

$$2\, t^{-2n-1}u(t) = |t|^{-2n-1} + |t|^{-2n-1}\mathrm{sgn}(t) = t^{-2n-1}\mathrm{sgn}(t) + t^{-2n-1}$$

and

$$FT[t^{-2n-1}u(t)] = \frac{(-1)^n\omega^{2n}}{(2n)!}[\log|\omega| + \gamma] - j\frac{(-1)^n\pi}{(2n)!}|\,\omega\,|^{2n}\mathrm{sgn}(\omega) \quad (51)$$

Let us return back to equation (11) and rewrite it in the format:

$$FT[t^{-\alpha-1}u(t)] = \Gamma(-\alpha)\,|\omega|^{\alpha}e^{j\alpha\pi/2\,\mathrm{sgn}(\omega)} \quad (52)$$

and compute the order one derivative relatively to α in both sides of the equation to obtain

$$FT[-t^{-\alpha-1}u(t)\log(t)] = -\Gamma'(-\alpha)\,|\omega|^{\alpha}e^{j\alpha\pi/2\,\mathrm{sgn}(\omega)} +$$

$$\Gamma(-\alpha)\,|\omega|^{\alpha}\log(|\omega|)e^{j\alpha\pi/2\,\mathrm{sgn}(\omega)}$$

$$+\,\Gamma(-\alpha)\,|\omega|^{\alpha}e^{j\alpha\pi/2\,\mathrm{sgn}(\omega)}\,j\alpha\pi\delta(\omega) \quad (53)$$

With α=-1, we obtain

$$FT[\log(t)u(t)] = -j\Gamma'(1)\,|\omega|^{-1}\mathrm{sgn}(\omega) - j\,|\omega|^{-1}\log(|\omega|) +$$

$$+\,\pi|\omega|^{-1}\mathrm{sgn}(\omega)\delta(\omega) \quad (54)$$

and

$$FT[\log(t)u(t)] = \pi\omega^{-1}\delta(\omega) + j|\omega|^{-1}[\gamma\mathrm{sgn}(\omega) - \log(|\omega|)] \quad (55)$$

But (Hoskins and Pinto, 1994)

$$\omega^{-1}\delta(\omega) = -\tfrac{1}{2}\delta^{(1)}(\omega) \quad (56)$$

that leads to

$$FT[\log(t)u(t)] = -\tfrac{1}{2}\delta^{(1)}(\omega) + j|\omega|^{-1}[\gamma\mathrm{sgn}(\omega) - \log(|\omega|)] \quad (57)$$

a somewhat strange distribution.

5. AN APPLICATION TO THE ALPHA STABLE DISTRIBUTION

We are going to consider the symmetric alpha stable distribution. This function is defined through its characteristic function

$$\Phi(\omega) = e^{-|\omega|^{\alpha}} \quad (58)$$

To obtain the corresponding probability density, we must compute the inverse Fourier transform of (46). There are two series that represent the probability density for $\alpha < 1$ and for the case $1 \le \alpha \le 2$. We are going to use the above results to obtain one of them. We start with the series obtained from the McLaurin series of the exponential.

$$\Phi(\omega) = e^{-|\omega|^{\alpha}} = \sum_{0}^{\infty}\frac{(-1)^n}{n!}|\omega|^{n\alpha} \quad (59)$$

Now, we are going to invert the series term by term. In order to complete the inversion, we are going to use the above results. We must be careful with the terms corresponding to integer values of $n\alpha$. For the cases with $n\alpha$ not integer, we have no problems. We only have to use the duality:

$$FT^{-1}[\,|\omega|^{n\alpha}\,] = -\frac{1}{\pi}\Gamma(n\alpha+1)\sin(n\alpha\pi/2)\,|\,t\,|^{-n\alpha-1} \quad (60)$$

In $n\alpha$ is an even positive integer, the inverse is a derivative of the delta and so it is zero for $t\neq 0$. Thus, the corresponding terms will be discarded. If $n\alpha$ is an odd positive integer, we write the condition, $n\alpha = 2k+1$ and use (34) to obtain:

$$FT^{-1}[\,|\omega|^{2k+1}\,] = -\frac{1}{\pi}\Gamma(2k+2)|\,t\,|^{-2k-2} \quad (61)$$

So, the inverse function is given by:

$$\varphi(t) = -\frac{1}{\pi}\sum_{0}^{\infty}\frac{(-1)^n}{n!}\Gamma(n\alpha+1)\sin(n\alpha\pi/2)\,|\,t\,|^{-n\alpha-1} \quad (62)$$

This series was presented by Uchaikin and Zolotarev (1999) that pointed out that it is convergent for $0 < \alpha < 1$, but nothing is said about the even order terms. In practice, the convergence of this series is very difficult as soon as t approaches the origin, becoming divergent. With a similar procedure it is a simple task to obtain the inverse of the general case:

$$\Phi(\omega) = e^{-\gamma|\omega|^{\alpha}e^{j\rho\pi/2\,\mathrm{sgn}(\omega)}} \quad (63)$$

but we do not have the space to complete this case here.

6. ACKNOWLEDGEMENT

The work of the first author was supported by FCT (CTS multiannual funding) through the PIDDAC Program funds.

The work of the second author is supported, in part, by the NIH, NIBIB grant, R 01 EB 007537.

7. CONCLUSIONS

The permanent presence of the power functions in fractional calculus motivated us to compute their Fourier transforms. This was done with the help of the distribution theory. We deduced all the transforms. Some of the obtained results are not available in current literature or are in contradiction with it. A table with all the transforms was constructed.

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APPENDIX – TABLE OF POWERS AND THEIR FOURIER TRANSFORMS

Function	Fourier Transform
$\frac{t^{-\alpha-1}u(t)}{\Gamma(-\alpha)}$	$\lvert\omega\rvert^{\alpha}.e^{j\alpha\pi/2 sgn(\omega)}$
$t^{2n} = \lvert t \rvert^{2n}$	$2\pi(-1)^n \delta^{(2n)}(\omega)$
$\lvert t \rvert^{2n+1}sgn(t)$	$2\pi(-1)^{n+1} j\delta^{(2n+1)}(\omega)$
$\lvert t \rvert^{\alpha}$ $\alpha\neq-2N$, $N\in\mathbf{N_0}$	$-2\Gamma(\alpha+1)\sin(\alpha\pi/2) \lvert \omega \rvert^{-\alpha-1}$
$\lvert t \rvert^{\alpha} sgn(t)$ $\alpha\neq-2N-1$, $N\in\mathbf{N_0}$	$-j2\Gamma(\alpha+1)\cos(\alpha\pi/2) \lvert \omega \rvert^{-\alpha-1} sgn(\omega)$
$\lvert t\rvert^{-\alpha-1}$	$2\Gamma(-\alpha)\cos(\alpha\pi/2)\lvert \omega \rvert^{\alpha}$
$\lvert t\rvert^{-\alpha-1}sgn(t)$	$2j\Gamma(-\alpha)\sin(\alpha\pi/2)\lvert \omega \rvert^{\alpha} sgn(\omega)$
$\lvert t \rvert^{2n+1}$	$-2\frac{(2n+1)!}{\omega^{2n+2}}$
$\lvert t \rvert^{2n} sgn(t)$	$2j\frac{(2n)!}{\omega^{2n+1}}$
$\lvert t \rvert^{-2n} = t^{-2n}$	$\pi\frac{(-1)^n \lvert\omega\rvert^{2n-1}}{(2n-1)!}$
$\lvert t \rvert^{-2n}sgn(t) = t^{-2n-1}$	$-\pi j\frac{(-1)^n \omega^{2n}sgn(\omega)}{(2n)!}$
$\lvert t \rvert^{-2n-1}$	$\frac{2(-1)^n\omega^{2n}}{(2n)!}[\log\lvert\omega\rvert + \gamma]$
FT[$\lvert t\rvert^{-2n-1}sgn(t)$]	$j\frac{(-1)^n\pi}{(2n)!}\lvert \omega \rvert^{2n}sgn(\omega)$
$t^{-2n}u(t)$	$\frac{(-1)^n\pi}{2(2n-1)!}\lvert \omega \rvert^{2n-1} + \frac{j(-1)^{n-1}\omega^{2n-1}}{(2n)!}[\log\lvert\omega\rvert + \gamma]$
$t^{-2n-1}u(t)$	$\frac{(-1)^n\omega^{2n}}{(2n)!}[\log\lvert\omega\rvert + \gamma] - j\frac{(-1)^n\pi}{(2n)!}\lvert \omega \rvert^{2n}sgn(\omega)$
$\log(t)u(t)$	$-½.\delta'(\omega) + j\lvert\omega\rvert^{-1} [\gamma\, sgn(\omega) - \log(\lvert\omega\rvert)]$