Fractional Analogous Models in Mechanics and Gravity Theories

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Abstract: We briefly review our recent results on the geometry of nonholonomic manifolds and Lagrange–Finsler spaces and fractional calculus with Caputo derivatives. Such constructions are used for elaborating analogous models of fractional gravity and fractional Lagrange mechanics.

Keywords: fractional calculus, fractional geometry, analogous models, fractional gravity, fractional Lagrange–Finsler space.

1. INTRODUCTION

We can construct analogous fractional models of geometries and physical theories in explicit form if we use fractional derivatives resulting in zero for actions on constants (for instance, for the Caputo fractional derivative). This is important for elaborating geometric models of theories with fractional calculus even (performing corresponding nonholonomic deformations) we may prefer to work with another type of fractional derivatives.

In this paper, we outline some key constructions for analogous classical and quantum fractional theories (1; 2; 3; 4; 5; 6) when methods of nonholonomic and Lagrange–Finsler geometry are generalized to fractional dimensions. [1]

An important consequence of such geometric approaches is that using analogous and bi–Hamilton models (see integer dimension constructions (7; 9; 10)) and related solitonic systems we can study analytically and numerically, as well to try to construct some analogous mechanical and gravitational systems, with the aim to mimic a nonlinear/fractional nonholonomic dynamics/evolution and even to provide certain schemes of quantization, like in the "fractional" Fedosov approach (4; 8).

This work is organized in the form: In section 2, we remember the most important formulas on Caputo fractional derivatives and nonlinear connections. Section 3 is devoted to fractional Lagrange–Finsler geometries. There are presented the main constructions for analogous fractional gravity in section 4.

[1] we recommend readers to consult in advance the above cited papers on details, notation conventions and bibliography

2. CAPUTO FRACTIONAL DERIVATIVES AND N–CONNECTIONS

We summarize some important formulas on fractional calculus for nonholonomic manifold elaborated in Refs. (1; 2; 3; 5). Our geometric arena consists from an abstract fractional manifold $\ {}^{\alpha}\mathbf{V}$ (we shall use also the term "fractional space" as an equivalent one enabled with certain fundamental geometric structures) with prescribed nonholonomic distribution modeling both the fractional calculus and the non–integrable dynamics of interactions.

The fractional left, respectively, right Caputo derivatives are denoted in the form

$$
{}_{\ 1x}\ {}^{\alpha}\underline{\partial}_{x} f(x) := \frac{\int\limits_{\ 1x}^{x} (x-\ x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^{s} f(x')dx'}{\Gamma(s-\alpha)}; \quad (1)
$$

$$
{}_{\ x}\ {}^{\alpha}\underline{\partial}\ {}_{2x} f(x) := \frac{1}{\Gamma(s-\alpha)} \int\limits_{x}^{2x} (x'-x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^{s} \times
$$

$$
f(x')dx'\ .
$$

Using such operators, we can construct the fractional absolute differential $\ {}^{\alpha}d := (dx^j)^{\alpha}\ \ {}_{0}\ {}^{\alpha}\underline{\partial}_{j}$ when $\ {}^{\alpha}dx^j = (dx^j)^{\alpha}\frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}$, where we consider $\ {}_{1}x^i = 0$.

We denote a fractional tangent bundle in the form $\ {}^{\alpha}\underline{T}M$ for $\alpha \in (0,1)$, associated to a manifold M of necessary smooth class and integer $\dim M = n$. [2] Locally, both the integer and fractional local coordinates are written in the form $u^{\beta} = (x^j, y^a)$. A fractional frame basis $\ {}^{\alpha}\underline{e}_{\beta} = e^{\beta'}_{\ \beta}(u^{\beta})\ {}^{\alpha}\underline{\partial}_{\beta'}$ on $\ {}^{\alpha}\underline{T}M$ is connected via a vierlbein transform $e^{\beta'}_{\ \beta}(u^{\beta})$ with a fractional local coordinate basis

[2] The symbol T is underlined in order to emphasize that we shall associate the approach to a fractional Caputo derivative.

$$ {}^{\alpha}\underline{\partial}_{\beta'} = \left({}^{\alpha}\underline{\partial}_{j'} = {}_{1x^{j'}}\ {}^{\alpha}\underline{\partial}_{j'},\ {}^{\alpha}\underline{\partial}_{b'} = {}_{1y^{b'}}\ {}^{\alpha}\underline{\partial}_{b'} \right), \quad (2) $$

for $j' = 1, 2, ..., n$ and $b' = n+1, n+2, ..., n+n$. The fractional co–bases are written ${}^{\alpha}\underline{e}^{\ \beta} = e^{\ \beta}_{\beta'}(u^{\beta})\ {}^{\alpha}du^{\beta'}$, where the fractional local coordinate co–basis is

$$ {}^{\alpha}du^{\beta'} = \left((dx^{i'})^{\alpha}, (dy^{a'})^{\alpha} \right). \quad (3) $$

It is possible to define a nonlinear connection (N–connection) ${}^{\alpha}\mathbf{N}$ for a fractional space ${}^{\alpha}\mathbf{V}$ by a nonholonomic distribution (Whitney sum) with conventional h– and v–subspaces, $\underline{h}\ {}^{\alpha}\mathbf{V}$ and $\underline{v}\ {}^{\alpha}\mathbf{V}$,

$$ {}^{\alpha}\underline{T}\ {}^{\alpha}\mathbf{V} = \underline{h}\ {}^{\alpha}\mathbf{V} \oplus \underline{v}\ {}^{\alpha}\mathbf{V}. \quad (4) $$

Locally, such a fractional N–connection is characterized by its local coefficients ${}^{\alpha}\mathbf{N} = \{ {}^{\alpha}N^{a}_{i} \}$, when ${}^{\alpha}\mathbf{N} = {}^{\alpha}N^{a}_{i}(u)(dx^{i})^{\alpha} \otimes {}^{\alpha}\underline{\partial}_{a}$.

On ${}^{\alpha}\mathbf{V}$, it is convenient to work with N–adapted fractional (co) frames,

$$ {}^{\alpha}\mathbf{e}_{\beta} = \left[{}^{\alpha}\mathbf{e}_{j} = {}^{\alpha}\underline{\partial}_{j} - {}^{\alpha}N^{a}_{j}\ {}^{\alpha}\underline{\partial}_{a},\ {}^{\alpha}e_{b} = {}^{\alpha}\underline{\partial}_{b} \right], \quad (5) $$
$$ {}^{\alpha}\mathbf{e}^{\beta} = [\ {}^{\alpha}e^{j} = (dx^{j})^{\alpha},\ {}^{\alpha}\mathbf{e}^{b} = (dy^{b})^{\alpha} + {}^{\alpha}N^{b}_{k}(dx^{k})^{\alpha}]. \quad (6) $$

A fractional metric structure (d–metric) ${}^{\alpha}\mathbf{g} = \{ {}^{\alpha}g_{\underline{\alpha}\underline{\beta}} \} = [\ {}^{\alpha}g_{kj},\ {}^{\alpha}g_{cb}]$ on ${}^{\alpha}\mathbf{V}$ can be represented in different equivalent forms,

$$ \begin{aligned} {}^{\alpha}\mathbf{g} &= {}^{\alpha}g_{\underline{\gamma}\underline{\beta}}(u)(du^{\underline{\gamma}})^{\alpha} \otimes (du^{\underline{\beta}})^{\alpha} \quad (7) \\ &= {}^{\alpha}g_{kj}(x,y)\ {}^{\alpha}e^{k} \otimes {}^{\alpha}e^{j} + {}^{\alpha}g_{cb}(x,y)\ {}^{\alpha}\mathbf{e}^{c} \otimes {}^{\alpha}\mathbf{e}^{b} \\ &= \eta_{k'j'}\ {}^{\alpha}e^{k'} \otimes {}^{\alpha}e^{j'} + \eta_{c'b'}\ {}^{\alpha}\mathbf{e}^{c'} \otimes {}^{\alpha}\mathbf{e}^{b'}, \end{aligned} $$

where matrices $\eta_{k'j'} = diag[\pm1, \pm1, ..., \pm1]$ and $\eta_{a'b'} = diag[\pm1, \pm1, ..., \pm1]$, for the signature of a "prime" spacetime $\mathbf{V}$, are obtained by frame transforms $\eta_{k'j'} = e^{k}_{\ k'}\ e^{j}_{\ j'}\ {}^{\alpha}g_{kj}$ and $\eta_{a'b'} = e^{a}_{\ a'}\ e^{b}_{\ b'}\ {}^{\alpha}g_{ab}$.

We can adapt geometric objects on ${}^{\alpha}\mathbf{V}$ with respect to a given N–connection structure ${}^{\alpha}\mathbf{N}$, calling them as distinguished objects (d–objects). For instance, a distinguished connection (d–connection) ${}^{\alpha}\mathbf{D}$ on ${}^{\alpha}\mathbf{V}$ is defined as a linear connection preserving under parallel transports the Whitney sum (4). There is an associated N–adapted differential 1–form

$$ {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta} = {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta\gamma}\ {}^{\alpha}\mathbf{e}^{\gamma}, \quad (8) $$

parametrizing the coefficients (with respect to (6) and (5)) in the form ${}^{\alpha}\mathbf{\Gamma}^{\gamma}_{\ \tau\beta} = \left({}^{\alpha}L^{i}_{jk},\ {}^{\alpha}L^{a}_{bk},\ {}^{\alpha}C^{i}_{jc},\ {}^{\alpha}C^{a}_{bc} \right)$.

The absolute fractional differential ${}^{\alpha}\mathbf{d} = {}_{1x}\ {}^{\alpha}d_{x} + {}_{1y}\ {}^{\alpha}d_{y}$ acts on fractional differential forms in N–adapted form. This is a fractional distinguished operator, d–operator, when the value ${}^{\alpha}\mathbf{d} := {}^{\alpha}\mathbf{e}^{\beta}\ {}^{\alpha}\mathbf{e}_{\beta}$ splits into exterior h- and v–derivatives when

$$ {}_{1x}\ {}^{\alpha}d_{x} := (dx^{i})^{\alpha} \quad {}_{1x}\ {}^{\alpha}\underline{\partial}_{i} = {}^{\alpha}e^{j}\ {}^{\alpha}\mathbf{e}_{j} \quad (9) $$

and

$$ {}_{1y}\ {}^{\alpha}d_{y} := (dy^{a})^{\alpha} \quad {}_{1x}\ {}^{\alpha}\underline{\partial}_{a} = {}^{\alpha}\mathbf{e}^{b}\ {}^{\alpha}e_{b}. \quad (10) $$

Using such differentials, we can compute in explicit form the torsion and curvature (as fractional two d–forms derived for (8)) of a fractional d–connection ${}^{\alpha}\mathbf{D} = \{ {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta\gamma} \}$,

$$ \begin{aligned} {}^{\alpha}\mathcal{T}^{\tau} &= {}^{\alpha}\mathbf{D}\ {}^{\alpha}\mathbf{e}^{\tau} = {}^{\alpha}\mathbf{d}\ {}^{\alpha}\mathbf{e}^{\tau} + {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta} \wedge {}^{\alpha}\mathbf{e}^{\beta} \text{ and} \quad (11) \\ {}^{\alpha}\mathcal{R}^{\tau}_{\ \beta} &= {}^{\alpha}\mathbf{D}\ {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta} \\ &= {}^{\alpha}\mathbf{d}\ {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \beta} - {}^{\alpha}\mathbf{\Gamma}^{\gamma}_{\ \beta} \wedge {}^{\alpha}\mathbf{\Gamma}^{\tau}_{\ \gamma} = {}^{\alpha}\mathbf{R}^{\tau}_{\ \beta\gamma\delta}\ {}^{\alpha}\mathbf{e}^{\gamma} \wedge {}^{\alpha}\mathbf{e}^{\delta}. \end{aligned} $$

Contracting respectively the indices, we can compute the fractional Ricci tensor ${}^{\alpha}\mathcal{R}ic = \{ {}^{\alpha}\mathbf{R}_{\alpha\beta} = {}^{\alpha}\mathbf{R}^{\tau}_{\ \alpha\beta\tau} \}$ with components

$$ \begin{aligned} {}^{\alpha}R_{ij} &= {}^{\alpha}R^{k}_{\ ijk}, \quad {}^{\alpha}R_{ia} = -\ {}^{\alpha}R^{k}_{\ ika}, \\ {}^{\alpha}R_{ai} &= {}^{\alpha}R^{b}_{\ aib}, \quad {}^{\alpha}R_{ab} = {}^{\alpha}R^{c}_{\ abc} \quad (12) \end{aligned} $$

and the scalar curvature of fractional d–connection ${}^{\alpha}\mathbf{D}$,

$$ \begin{aligned} {}^{\alpha}_{s}\mathbf{R} &= {}^{\alpha}\mathbf{g}^{\tau\beta}\ {}^{\alpha}\mathbf{R}_{\tau\beta} = {}^{\alpha}R + {}^{\alpha}S, \\ {}^{\alpha}R &= {}^{\alpha}g^{ij}\ {}^{\alpha}R_{ij}, \quad {}^{\alpha}S = {}^{\alpha}g^{ab}\ {}^{\alpha}R_{ab}, \quad (13) \end{aligned} $$

with ${}^{\alpha}\mathbf{g}^{\tau\beta}$ being the inverse coefficients to a d–metric (7).

The Einstein tensor of any metric compatible ${}^{\alpha}\mathbf{D}$, when ${}^{\alpha}\mathbf{D}_{\tau}\ {}^{\alpha}\mathbf{g}^{\tau\beta} = 0$, is defined ${}^{\alpha}\mathcal{E}ns = \{ {}^{\alpha}\mathbf{G}_{\alpha\beta} \}$, where

$$ {}^{\alpha}\mathbf{G}_{\alpha\beta} := {}^{\alpha}\mathbf{R}_{\alpha\beta} - \frac{1}{2}\ {}^{\alpha}\mathbf{g}_{\alpha\beta}\ {}^{\alpha}_{s}\mathbf{R}. \quad (14) $$

The regular fractional mechanics defined by a fractional Lagrangian ${}^{\alpha}L$ can be equivalently encoded into canonical geometric data $(\ {}_{L}\ {}^{\alpha}\mathbf{N},\ {}_{L}\ {}^{\alpha}\mathbf{g},\ {}^{\alpha}_{c}\mathbf{D})$, where we put the label L in order to emphasize that such geometric objects are induced by a fractional Lagrangian as we provided in (1; 2; 3; 5). We also note that it is possible to "arrange" on ${}^{\alpha}\mathbf{V}$ such nonholonomic distributions when a d–connection ${}_{0}\ {}^{\alpha}\mathbf{D} = \{ {}^{\alpha}_{0}\widehat{\mathbf{\Gamma}}^{\gamma'}_{\ \alpha'\beta'} \}$ is described by constant matrix coefficients, see details in (9; 10), for integer dimensions, and (5), for fractional dimensions.

3. FRACTIONAL LAGRANGE–FINSLER GEOMETRY

A Lagrange space $L^{n} = (M, L)$, of integer dimension n, is defined by a Lagrange fundamental function $L(x,y)$, i.e. a regular real function $L : TM \to R$, for which the Hessian ${}_{L}g_{ij} = (1/2)\partial^{2}L/\partial y^{i}\partial y^{j}$ is not degenerate.

We say that a Lagrange space L^{n} is a Finsler space F^{n} if and only if its fundamental function L is positive and two homogeneous with respect to variables y^{i}, i.e. $L = F^{2}$. For simplicity, we shall work with Lagrange spaces and their fractional generalizations, considering the Finsler ones to consist of a more particular, homogeneous, subclass.

Definition: A (target) fractional Lagrange space ${}^{\alpha}\underline{L^{n}} = (\ {}^{\alpha}\underline{M},\ {}^{\alpha}L)$ of fractional dimension $\alpha \in (0,1)$, for a regular real function ${}^{\alpha}L :\ {}^{\alpha}\underline{T}M \to R$, when the fractional Hessian is

$$ {}_{L}\ {}^{\alpha}g_{ij} = \frac{1}{4}\left({}^{\alpha}\underline{\partial}_{i}\ {}^{\alpha}\underline{\partial}_{j} + {}^{\alpha}\underline{\partial}_{j}\ {}^{\alpha}\underline{\partial}_{i} \right)\ {}^{\alpha}L \neq 0. \quad (15) $$

In our further constructions, we shall use the coefficients ${}_{L}\ {}^{\alpha}g^{ij}$being inverse to ${}_{L}\ {}^{\alpha}g_{ij}$ (15).[3] Any ${}^{\alpha}\underline{L^{n}}$ can be associated to a prime "integer" Lagrange space L^{n}.

The concept of nonlinear connection (N–connection) on ${}^{\alpha}\underline{L^{n}}$ can be introduced similarly to that on nonholonomic fractional manifold (1; 2) considering the fractional tangent bundle ${}^{\alpha}\underline{T}M$.

Definition: A N–connection ${}^{\alpha}\mathbf{N}$ on ${}^{\alpha}\underline{T}M$ is defined by a nonholonomic distribution (Whitney sum) with conventional h– and v–subspaces, $\underline{h}\ {}^{\alpha}\underline{T}M$ and $\underline{v}\ {}^{\alpha}\underline{T}M$, when

$$ {}^{\alpha}\underline{T}\ {}^{\alpha}\underline{T}M=\underline{h}\ {}^{\alpha}\underline{T}M\oplus\underline{v}\ {}^{\alpha}\underline{T}M. \qquad (16)$$

Locally, a fractional N–connection is defined by a set of coefficients, ${}^{\alpha}\mathbf{N}$=$\{\ {}^{\alpha}N_{i}^{a}\}$, when

$$ {}^{\alpha}\mathbf{N}=\ {}^{\alpha}N_{i}^{a}(u)(dx^{i})^{\alpha}\otimes\ {}^{\alpha}\underline{\partial}_{a}, \qquad (17)$$

see local bases (2) and (3).

Let us consider values $y^{k}(\tau)\ =\ dx^{k}(\tau)/d\tau$, for $x(\tau)$ parametrizing smooth curves on a manifold M with $\tau\in[0,1]$. The fractional analogs of such configurations are determined by changing $d/d\tau$ into the fractional Caputo derivative ${}^{\alpha}\underline{\partial}_{\tau}={}_{1\tau}\ {}^{\alpha}\underline{\partial}_{\tau}$ when ${}^{\alpha}y^{k}(\tau)=\ {}^{\alpha}\underline{\partial}_{\tau}x^{k}(\tau)$. For simplicity, we shall omit the label α for $y\in\ {}^{\alpha}\underline{T}M$ if that will not result in ambiguities and/or we shall do not associate to it an explicit fractional derivative along a curve.

By straightforward computations, following the same scheme as in (7) but with fractional derivatives and integrals, we prove:

Theorem: Any ${}^{\alpha}L$ defines the fundamental geometric objects determining canonically a nonholonomic fractional Riemann–Cartan geometry on ${}^{\alpha}\underline{T}M$ being satisfied the properties:

(1) The fractional Euler–Lagrange equations

$$ {}^{\alpha}\underline{\partial}_{\tau}\ (\ {}_{1}{}_{y^{i}}\ {}^{\alpha}\underline{\partial}_{i}\ {}^{\alpha}L)-{}_{1}{}_{x^{i}}\ {}^{\alpha}\underline{\partial}_{i}\ {}^{\alpha}L=0 \qquad (18)$$

are equivalent to the fractional "nonlinear geodesic" (equivalently, semi–spray) equations

$$ (\ {}^{\alpha}\underline{\partial}_{\tau}\ )^{2}\,x^{k}+2\ {}^{\alpha}G^{k}(x,\ {}^{\alpha}y)=0, \qquad (19)$$

where

$$ {}^{\alpha}G^{k}=\frac{1}{4}\ {}_{L}\ {}^{\alpha}g^{kj}[y^{j}\ \ {}_{1}{}_{y^{j}}\ {}^{\alpha}\underline{\partial}_{j}\ \ (\ {}_{1}{}_{x^{i}}\ {}^{\alpha}\underline{\partial}_{i}\ {}^{\alpha}L)- {}_{1}{}_{x^{i}}\ {}^{\alpha}\underline{\partial}_{i}\ {}^{\alpha}L]$$

defines the canonical N–connection

$$ {}_{L}^{\alpha}N_{j}^{a}=\ {}_{1}{}_{y^{j}}\ {}^{\alpha}\underline{\partial}_{j}\ {}^{\alpha}G^{k}(x,\ {}^{\alpha}y). \qquad (20)$$

(2) There is a canonical (Sasaki type) metric structure,

$$ {}_{L}\ {}^{\alpha}\mathbf{g}=\ {}_{L}^{\alpha}g_{kj}(x,y)\ {}^{\alpha}e^{k}\otimes\ {}^{\alpha}e^{j}+ {}_{L}^{\alpha}g_{cb}(x,y)\ {}_{L}^{\alpha}\mathbf{e}^{c}\otimes\ {}_{L}^{\alpha}\mathbf{e}^{b},$$

where the preferred frame structure (defined linearly by ${}_{L}^{\alpha}N_{j}^{a}$) is ${}_{L}^{\alpha}\mathbf{e}_{\nu}=(\ {}_{L}^{\alpha}\mathbf{e}_{i},e_{a})$.

(3) There is a canonical metrical distinguished connection

$$ {}_{c}^{\alpha}\mathbf{D}=(h\ {}_{c}^{\alpha}D,v\ {}_{c}^{\alpha}D)=\{\ {}_{c}^{\alpha}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} =(\ {}^{\alpha}\widehat{L}^{i}{}_{jk},\ {}^{\alpha}\widehat{C}_{jc}^{i})\}\ ,$$

(in brief, d–connection), which is a linear connection preserving under parallelism the splitting (16) and metric compatible, i.e. ${}_{c}^{\alpha}\mathbf{D}\ (\ {}_{L}\ {}^{\alpha}\mathbf{g})=0$, when

$$ {}_{c}^{\alpha}\mathbf{\Gamma}^{i}{}_{j}=\ {}_{c}^{\alpha}\mathbf{\Gamma}^{i}{}_{j\gamma}\ {}_{L}^{\alpha}\mathbf{e}^{\gamma}=\widehat{L}^{i}{}_{jk}e^{k}+\widehat{C}_{jc}^{i}\ {}_{L}^{\alpha}\mathbf{e}^{c}, \qquad (21)$$

for $\widehat{L}^{i}{}_{jk}=\widehat{L}^{a}{}_{bk},\widehat{C}_{jc}^{i}=\widehat{C}_{bc}^{a}$ in ${}_{c}^{\alpha}\mathbf{\Gamma}^{a}{}_{b}=\ {}_{c}^{\alpha}\mathbf{\Gamma}^{a}{}_{b\gamma}\ {}_{L}^{\alpha}\mathbf{e}^{\gamma}=\widehat{L}^{a}{}_{bk}e^{k}+\widehat{C}_{bc}^{a}\ {}_{L}^{\alpha}\mathbf{e}^{c}$,

$$ {}^{\alpha}\widehat{L}_{jk}^{i}=\frac{1}{2}\ {}_{L}^{\alpha}g^{ir}\ (\ {}_{L}^{\alpha}\mathbf{e}_{k}\ {}_{L}^{\alpha}g_{jr}+\ {}_{L}^{\alpha}\mathbf{e}_{j}\ {}_{L}^{\alpha}g_{kr}-\ {}_{L}^{\alpha}\mathbf{e}_{r}\ {}_{L}^{\alpha}g_{jk})\,,$$

$$ {}^{\alpha}\widehat{C}_{bc}^{a}=\frac{1}{2}\ {}_{L}^{\alpha}g^{ad}\ (\ {}^{\alpha}e_{c}\ {}_{L}^{\alpha}g_{bd}+\ {}^{\alpha}e_{c}\ {}_{L}^{\alpha}g_{cd}-\ {}^{\alpha}e_{d}\ {}_{L}^{\alpha}g_{bc})$$

are just the generalized Christoffel indices.[4]

Finally, in this section, we note that:

Remark: We note that ${}_{c}^{\alpha}\mathbf{D}$ is with nonholonomically induced torsion structure defined by 2–forms

$$ {}_{L}^{\alpha}\mathcal{T}^{i}=\widehat{C}^{i}{}_{jc}\ {}^{\alpha}e^{i}\wedge\ {}_{L}^{\alpha}\mathbf{e}^{c},$$
$$ {}_{L}^{\alpha}\mathcal{T}^{a}=-\frac{1}{2}\ {}_{L}\Omega_{ij}^{a}\ {}^{\alpha}e^{i}\wedge\ {}^{\alpha}e^{j}+ \left(\ {}^{\alpha}e_{b}\ {}_{L}^{\alpha}N_{i}^{a}-\ {}^{\alpha}\widehat{L}^{a}{}_{bi}\right)\ {}^{\alpha}e^{i}\wedge\ {}_{L}^{\alpha}\mathbf{e}^{b}$$

computed from the fractional version of Cartan's structure equations

$$ d\ {}^{\alpha}e^{i}-\ {}^{\alpha}e^{k}\wedge\ \ {}_{c}^{\alpha}\mathbf{\Gamma}^{i}{}_{k}=-\ {}_{L}^{\alpha}\mathcal{T}^{i},$$
$$ d\ {}_{L}^{\alpha}\mathbf{e}^{a}-\ {}_{L}^{\alpha}\mathbf{e}^{b}\wedge\ \ {}_{c}^{\alpha}\mathbf{\Gamma}^{a}{}_{b}=-\ {}_{L}^{\alpha}\mathcal{T}^{a},$$
$$ d\ \ {}_{c}^{\alpha}\mathbf{\Gamma}^{i}{}_{j}-\ \ {}_{c}^{\alpha}\mathbf{\Gamma}^{k}{}_{j}\wedge\ \ {}_{c}^{\alpha}\mathbf{\Gamma}^{i}{}_{k}=-\ {}_{L}^{\alpha}\mathcal{R}_{j}^{i}$$

in which the curvature 2–form is denoted ${}_{L}^{\alpha}\mathcal{R}_{j}^{i}$.

In general, for any d–connection on ${}^{\alpha}\underline{T}M$, we can compute respectively the N–adapted coefficients of torsion ${}^{\alpha}\mathcal{T}^{\tau}=\{\ {}^{\alpha}\mathbf{\Gamma}^{\tau}{}_{\beta\gamma}\}$ and curvature ${}^{\alpha}\mathcal{R}^{\tau}{}_{\beta}=\{\ {}^{\alpha}\mathbf{R}^{\tau}{}_{\beta\gamma\delta}\}$ as it is explained for general fractional nonholonomic manifolds in (1; 2).

4. ANALOGOUS FRACTIONAL GRAVITY

Let us consider a "prime" nonholonomic manifold $\mathbf{V}$ is of integer dimension dim $\mathbf{V}=n+m,n\geq2,m\geq1$.[5] Its fractional extension ${}^{\alpha}\mathbf{V}$ is modelled by a quadruple $(\mathbf{V},\ {}^{\alpha}\mathbf{N},\ {}^{\alpha}\mathbf{d},\ {}^{\alpha}\mathbf{I})$, where ${}^{\alpha}\mathbf{N}$ is a nonholonomic distribution stating a nonlinear connection (N–connection) structure. The fractional differential structure ${}^{\alpha}\mathbf{d}$ is determined

[3] We shall put a left label L to certain geometric objects if it is necessary to emphasize that they are induced by Lagrange generating function. Nevertheless, such labels will be omitted (in order to simplify the notations) if that will not result in ambiguities.

[4] for integer dimensions, we contract "horizontal" and "vertical" indices following the rule: $i=1$ is $a=n+1$; $i=2$ is $a=n+2$; ... $i=n$ is $a=n+n$"

[5] A nonholonomic manifold is a manifold endowed with a non–integrable (equivalently, nonholonomic, or anholonomic) distribution. There are three useful (for our considerations) examples when 1) $\mathbf{V}$ is a (pseudo) Riemannian manifold; 2) $\mathbf{V}=E(M)$, or 3) $\mathbf{V}=TM$, for a vector, or tangent, bundle on a base manifold M. We also emphasize that in this paper we follow the conventions from Refs. (7; 1; 2) when left indices are used as labels and right indices may be abstract ones or running certain values.

by Caputo fractional derivative (1) following formulas (2) and (3).

For any respective frame and co–frame (dual) structures, ${}^{\alpha}e_{\alpha'} = ({}^{\alpha}e_{i'}, {}^{\alpha}e_{a'})$ and ${}^{\alpha}e^{\beta'} = ({}^{\alpha}e^{i'}, {}^{\alpha}e^{a'})$ on ${}^{\alpha}\mathbf{V}$, we can consider frame transforms

$$ {}^{\alpha}e_{\alpha} = A_{\alpha}^{\ \alpha'}(x,y)\ {}^{\alpha}e_{\alpha'} \text{ and } {}^{\alpha}e^{\beta} = A^{\beta}_{\ \beta'}(x,y)\ {}^{\alpha}e^{\beta'}. \quad (22)$$

A subclass of frame transforms (22), for fixed "prime" and "target" frame structures, is called N–adapted if such nonholonomic transformations preserve the splitting defined by a N–connection structure $\mathbf{N} = \{N_i^a\}$.

Under (in general, nonholonomic) frame transforms, the metric coefficients of any metric structure ${}^{\alpha}\mathbf{g}$ on ${}^{\alpha}\mathbf{V}$ are re–computed following formulas

$$ {}^{\alpha}g_{\alpha\beta}(x,y) = A_{\alpha}^{\ \alpha'}(x,y)\ A_{\beta}^{\ \beta'}(x,y)\ {}^{\alpha}g_{\alpha'\beta'}(x,y). \quad (23)$$

For any fixed ${}^{\alpha}\mathbf{g}$ and ${}^{\alpha}\mathbf{N}$, there are N–adapted frame transforms when

$$ {}^{\alpha}\mathbf{g} = {}^{\alpha}g_{ij}(x,y)\ {}^{\alpha}e^{i} \otimes {}^{\alpha}e^{j} + {}^{\alpha}h_{ab}(x,y)\ {}^{\alpha}\mathbf{e}^{a} \otimes {}^{\alpha}\mathbf{e}^{b}, $$
$$ = {}^{\alpha}g_{i'j'}(x,y)\ {}^{\alpha}e^{i'} \otimes {}^{\alpha}e^{j'} + {}^{\alpha}h_{a'b'}(x,y)\ {}^{\alpha}\mathbf{e}^{a'} \otimes {}^{\alpha}\mathbf{e}^{b'}, $$

where ${}^{\alpha}\mathbf{e}^{a}$ and ${}^{\alpha}\mathbf{e}^{a'}$ are elongated following formulas (6), respectively by ${}^{\alpha}N_j^a$ and

$$ {}^{\alpha}N_{\ j'}^{a'} = A_{a}^{\ a'}(x,y) A^{j}_{\ j'}(x,y)\ {}^{\alpha}N_{\ j}^{a}(x,y), \quad (24)$$

or, inversely,

$$ {}^{\alpha}N_{\ j}^{a} = A_{a'}^{\ a}(x,y) A^{j'}_{\ j}(x,y)\ {}^{\alpha}N_{\ j'}^{a'}(x,y) \quad (25)$$

with prescribed ${}^{\alpha}N_{j'}^{a'}$.

We preserve the N–connection splitting for any frame transform of type (22) when

$$ {}^{\alpha}g_{i'j'} = A^{i}_{\ i'} A^{j}_{\ j'}\ {}^{\alpha}g_{ij}, \quad {}^{\alpha}h_{a'b'} = A^{a}_{\ a'} A^{b}_{\ b'}\ {}^{\alpha}h_{ab}, \quad (26)$$

for $A_i^{\ i'}$ constrained to get holonomic ${}^{\alpha}e^{i'} = A_i^{\ i'}\ {}^{\alpha}e^{i}$, i.e. $[{}^{\alpha}e^{i'}, {}^{\alpha}e^{j'}] = 0$ and ${}^{\alpha}\mathbf{e}^{a'} = dy^{a'} + {}^{\alpha}N^{a'}_{j'} dx^{j'}$, for certain $x^{i'} = x^{i'}(x^i, y^a)$ and $y^{a'} = y^{a'}(x^i, y^a)$, with ${}^{\alpha}N^{a'}_{j'}$ computed following formulas (24). Such conditions can be satisfied by prescribing from the very beginning a nonholonomic distribution of necessary type. The constructions can be equivalently inverted, when ${}^{\alpha}g_{\alpha\beta}$ and ${}^{\alpha}N_i^a$ are computed from ${}^{\alpha}g_{\alpha'\beta'}$ and ${}^{\alpha}N_{i'}^{a'}$, if both the metric and N–connection splitting structures are fixed on ${}^{\alpha}\mathbf{V}$.

An unified approach to Einstein–Lagrange/Finsler gravity for arbitrary integer and non–integer dimensions is possible for the fractional canonical d–connection ${}^{\alpha}\widehat{\mathbf{D}}$. The fractional gravitational field equations are formulated for the Einstein d–tensor (14), following the same principle of constructing the matter source ${}^{\alpha}\mathbf{\Upsilon}_{\beta\delta}$ as in general relativity but for fractional metrics and d–connections,

$$ {}^{\alpha}\widehat{\mathbf{E}}_{\ \beta\delta} = {}^{\alpha}\mathbf{\Upsilon}_{\beta\delta}. \quad (27)$$

Such a system of integro–differential equations for generalized connections can be restricted to fractional nonholonomic configurations for ${}^{\alpha}\nabla$ if we impose the additional constraints

$$ {}^{\alpha}\widehat{L}^{c}_{aj} = {}^{\alpha}e_a({}^{\alpha}N^c_j), \quad {}^{\alpha}\widehat{C}^{i}_{jb} = 0, \quad {}^{\alpha}\Omega^{a}_{\ ji} = 0. \quad (28)$$

There are not theoretical or experimental evidences that for fractional dimensions we must impose conditions of type (28) but they have certain physical motivation if we develop models which in integer limits result in the general relativity theory.

Acknowledgement: This paper summarizes the results presented in our talk at the 3d Conference on "Nonlinear Science and Complexity", 28–31 July, 2010, Çhankaya University, Ankara, Turkey.

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