An application of the discrete fractional variational principles

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Abstract: The fractional calculus is an emerging topic in the area of science and engineering. The fractional difference variational principles started to be investigated intensively during the last few years. In this paper we investigate the discrete fractional Euler-Lagrange of a free one-dimensional particle.

Keywords**:** fractional calculus,discrete formulation,variational principles,free particle

1. Introduction

The fractional calculus, which studies the calculus of derivatives and integrals of any order became of great importance in the last decade [1,2,3].

In [8] the authors defined the left and right fractional sums starting with linear difference equations. In [6,7], discrete fractional operators were studied and the commutativity properties of the fractional sum and fractional difference operators were developed. In [9], the right fractional sum and difference operators were defined and the integration by parts formula for fractional differences was developed.

We give now some basic definitions related to fractional differences calculus.

For natural number n, the factorial polynomial is defined by

$$t^{(n)} = \prod_{j=0}^{n-1}(t-j) = \frac{\Gamma(t+1)}{\Gamma(t+1-n)}. \qquad (1)$$

For arbitrary α, define

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}\ , \qquad (2)$$

where the convention $t^{(\alpha)} = 0$, when we divide over a pole.

If $\alpha > 0$ and $\sigma(s) = s+1$, the α-th left fractional sum of f is defined by

$$\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\sum_{n=\alpha}^{t-\alpha}(t-\sigma(s))^{(\alpha-1)} f(s)\,, \qquad (3)$$

and the α-th right fractional sum of f is defined by

$$\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\sum_{n=\alpha}^{t-\alpha}(\rho(s)-t))^{(\alpha-1)} f(s)\,, \qquad (4)$$

where $\rho(s) = s-1$.

The Riemann left and the Riemann right fractional differences are defined respectively by

$$\Delta^{\alpha} f(t) = \Delta^{n}\Delta^{-(n-\alpha)} f(s) \qquad (5)$$

$$\nabla^{\alpha} f(t) = (-1)^{n}\ \nabla^{n}\nabla^{-(n-\alpha)} f(s) \qquad (6)$$

This manuscript is organized as follows:
In section 2, we present the integration by parts formula. In section 3, the Euler-Lagrange equations are discussed. In section 4 an example is studied.

2. Integration by Parts for Fractional Differences

Proposition 1. Let $\alpha > 0$, a , b ∈ R such that a < b and b≡ a + α (mod 1). If f is defined on N_a and g is defined on ${}_bN$, then we have

$$\sum_{s=a+\alpha}^{b}(\Delta^{-\alpha} f)(s)g(s) = \sum_{s=a}^{b-a} f(s)\nabla^{-\alpha} g(s)\,, \qquad (7)$$

where $N_a = \{a, a+1, a+2, ...\}$ and ${}_bN = \{b, b-1, b-2, ...\}$.

Proposition 2. Let α >0 be non-integer and assume that b≡ $a+(n-\alpha)$ (mod 1). If f is defined on ${}_bN$ and g is defined on N_a , then

$$\sum_{s=a+(n-\alpha)-1}^{b-n+1} f(s)\Delta^{\alpha} g(s) = \sum_{s=a+n-1}^{b-(n-\alpha)+1} g(s)\nabla^{\alpha} f(s) \qquad (8)$$

When α =1 we obtain

$$\sum_{s=a}^{b-1} f(s)\Delta g(s)$$

$$= f(s)g(s)\Big|_a^b - \sum_{s=a+1}^{b} g(s)\nabla f(s) = f(s)g(s)\Big|_a^b - \sum_{s=a}^{b-1} g(s+1)\Delta f(s) \quad (9)$$

3. Euler-Lagrange equations

The obtained by-parts formula in previous section is very useful in discrete fractional calculus. Here, for example, we show how it is used to obtain Euler-Langrange equations for a discrete variational problem in fractional calculus. We consider the functional J: S $\longrightarrow$ R,

$$J(y) = \sum_{s=a-\alpha}^{b} L(s, y(s), \Delta^{\alpha} y(s)) \qquad (10)$$

where

a ,b $\in$ R, 0 < α < 1

L: $(N_{a-\alpha} \cap_{b+\alpha} N)$ x $(R^n)^2 \to R, b \equiv a+\alpha (\text{mod } 1)$ (11)

and

s $= \{y : N_{a-\alpha} \cap_{b+a} N \to R^{n:} y(a) = y_0 \;\; y(b+\alpha) = y_1\}$

Morover, we assume that the function y fits the discrete time scales N_a and $N_{a+n-\alpha}$. That is, $y(s) = y(s+n-\alpha)$ for all $s \in N_a$.

We shall shortly write:

L(s) $\equiv L(s, y(s), \Delta^a y(s))$ (12)

WE calculate the first varition of the functional J on the linear manifold S: Let $\eta \in H = \{h : \to R^n : h(a) = h(b+\alpha) = 0\}$, then

$$\delta J(y(x), (\eta(x) = \frac{d}{d\varepsilon} J(y(x) + \varepsilon \eta(x))\Big|_{\varepsilon=0}$$

$$\sum_{s=a-\alpha}^{b} [\frac{\partial L(s)}{\partial y} \eta(s) + \frac{\partial L(s)}{\partial \Delta^{\alpha} y} \Delta^{\alpha} \eta(s)]. \qquad (13)$$

Then we apply proposition with 0 < α < 1 and n=1 to get

$$\delta J(y(x), \eta(x)) = \sum_{s=a}^{b+\alpha} [\frac{\partial L(s)}{\partial y} + \nabla^{\alpha} \frac{\partial L(s)}{\partial \Delta^{\alpha} y}] \eta(s) = 0, \quad (14)$$

and by applying a suitable discrete fundamental lemma in calculus of variations we obtain the Euler-Lagrange equation:

$$\frac{\partial L(s)}{\partial y} + \nabla^{\alpha} \frac{\partial L(s)}{\partial \Delta^{\alpha} y} = 0. \qquad (15)$$

4. Example

We consider the following action

$$J(y) = \sum_{s=a-\alpha}^{b} \frac{1}{2} (\Delta^{\alpha} y(s))^2 \quad , \qquad (16)$$

where a ,*b* ∈ *R*, *0* < α < 1. Applying (15), we have

$$\nabla^{\alpha} (\Delta^{\alpha} y(s)) = 0. \qquad (17)$$

We should mention that, equation (17) is highly difficult to solve.

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