Strict Stability Criteria of Fractional order Differential Equations in Caputo's Sense in Terms of Two Measures with Initial Time Difference ⋆

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Abstract: In this article, we have investigated that the strict stability criteria of fractional order differential equations in terms of two measures with initial time difference for a perturbed fractional order differential system with respect to an unperturbed fractional order differential system that differs both in initial time and initial position.

Keywords: Strict stability in terms of two measures, initial time difference, fractional order differential equations, Lyapunov's stability.

1. INTRODUCTION

The history of fractional order derivative was started by L' Hospital and Leibnitz. See Kilbas et al. (2006), Lakshmikantham et al. (2009), Podlubny (1999), Samko et al. (1993) The question of L' Hospital to Leibnitz gave a new theory mathematicians. Then, the mathematicians has developed the fractional calculus and the theory of fractional differential equations. The improvement of these studies are continued still and gained new results math and this results are also applied different disciplines such as physics, chemistry and engineering. See Lakshmikantham et al. (2009), Lakshmikantham et al. (1999), Yakar (2010).

The application of Lyapunov's second method in stability of differential equations, please see Lakshmikantham et al. (1989), Lakshmikantham et al. (2009),Lakshmikantham et al. (2001), Shaw et al. (2000), Yakar (2010), Yakar (2007) and Yakar et al. (2005), that has the advantage of not requiring behavior of solutions of the system which investigates. Recently in the work of Yakar (2007), Yakar et al. (2008) and Yakar et al. (2009), the stability with initial time difference in terms of two measure and the properties of fractional differential equations in Lakshmikantham et al. (2009) and Yakar (2010) has been investigated. The strict stability criteria of differential equations in Lakshmikantham et al. (2009) and initial time difference strict stability worked and obtained comparison results and the appropriate definitions in the work of Yakar (2007).

In this paper, we investigated strict stability criteria with initial time difference in terms of two measure on fractional order differential equations and we have used the definition of Caputo's fractional order derivative because of some advantages which we express. We look into the strict stability criteria in terms of two measures with initial time difference a perturbed fractional order differential system with respect to an unperturbed fractional order differential system which have different initial time and initial position. Using Lyapunov functions and comparison principle have been given sufficient conditions for the strict stability of dynamic systems on fractional order differential equations.

2. PRELIMINARIES

In this study, we have used Caputo's fractional order derivative. But we have three definition of fractional order derivative: Caputo, Reimann-Liouville and Grünwald-Letnikov. The definition of Caputo's and Reimann-Liouville's fractional derivatives

$$^{c}D^{q}x=\frac{1}{\Gamma\left(1-q\right)}\int_{t_0}^{t}\left(t-s\right)^{-q}x'(s)ds, t_0\leq t\leq T \quad (2.1)$$

$$D^{q}x=\frac{1}{\Gamma\left(p\right)}\left(\frac{d}{dt}\int_{\tau_0}^{t}\left(t-s\right)^{p-1}x(s)ds\right), t_0\leq t\leq T \quad (2.2)$$

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order of $0 < q < 1$, and $p + q = 1$ where Γ denotes the Gamma function.

It can't be denied that the fractional derivative of Riemann-Liouville is importance for the development of fractional calculus and fractional order differential equations. But in mathematical modeling of some applications of various areas, there is a difficulty to interpret the initial condition required for the initial value problems of fractional order differential equations. The main advantage of Caputo's definition of fractional order derivative is that the initial conditions for fractional order differential equations with Caputo derivative take on the same form as that of ordinary differential equations with integer derivatives and another difference is that the Caputo derivative for a constant C is zero, while the Riemann-Liouville fractional derivative for a constant C is not zero but equals to $D^qC = \frac{C(t-t_0)^{-q}}{\Gamma(1-q)}$. By using (2.1) and therefore,

$$ {}^cD^q x(t) = D^q\left[x(t) - x(t_0)\right] \tag{2.3} $$

$$ {}^cD^q x(t) = D^q x(t) - \frac{x(t_0)}{\Gamma(1-q)}(t-t_0)^{-q}. \tag{2.4} $$

In particular, if $x(t_0) = 0$, the equality holds

$$ {}^cD^q x(t) = D^q x(t). \tag{2.5} $$

and Caputo's derivative is defined for functions for which Riemann-Liouville fractional order derivative exists.

Let us write that Grünwald-Letnikov's notion of fractional order derivative in a convenient form

$$ D_0^q x(t) = \lim_{\substack{h\to 0\\ nh=t-t_0}} \frac{1}{h^q}\left[x(t) - S(x,h,r,q)\right] \tag{2.6} $$

where $S(x,h,r,q) = \sum\limits_{r=1}^{n}(-1)^{r+1}\binom{q}{r}x(t-rh)$. If $x(t)$ is continuous and $\frac{dx(t)}{dt}$ exists and integrable, then Riemann-Liouville and Grünwald-Letnikov fractional order derivatives are connected by the relation

$$ D^q x(t) = D_0^q x(t) = \frac{x(t_0)(t-t_0)^{-q}}{\Gamma(1-q)} + \int_{\tau_0}^{t}\frac{(t-s)^{-q}}{\Gamma(1-q)}\frac{d}{ds}x(s)ds. \tag{2.7} $$

By using (2.3) implies that we have the relations among the Caputo, Riemann-Liouville and Grünwald-Letnikov fractional derivatives

$$ {}^cD^q x(t) = D^q\left[x(t) - x(t_0)\right] \tag{2.8} $$

$$ {}^cD^q x(t) = D_0^q\left[x(t) - x(t_0)\right] $$

$$ {}^cD^q x(t) = \frac{1}{\Gamma(1-q)}\int_{\tau_0}^{t}(t-s)^{-q}\frac{dx(s)}{ds}ds. $$

This relations of the definitions of the fractional order derivative are important to understand of the properties of the solutions of fractional order differential equations.

3. DEFINITION AND NOTATION

Consider the differential systems

$$ {}^cD^q x(t) = f(t,x), x(t_0) = x_0 \text{ for } t \geq t_0, t_0 \in \mathbb{R}_+ \tag{3.1} $$

$$ {}^cD^q y(t) = f(t,y), y(\tau_0) = y_0 \text{ for } t \geq \tau_0, \tau_0 \in \mathbb{R}_+ \tag{3.2} $$

where $x_0 = \lim_{t\to t_0} D^{q-1}x(t)$ and $y_0 = \lim_{t\to\tau_0} D^{q-1}y(t)$ exist and the perturbed fractional order differential system with Caputo's derivative of (3.1)

$$ {}^cD^q y(t) = F(t,y), y(\tau_0) = y_0 \text{ for } t \geq \tau_0 \tag{3.3} $$

where $y_0 = \lim_{t\to\tau_0} D^{q-1}y(t)$ exist and $f, F \in C[[t_0, \tau_0 + T] \times \mathbb{R}^n, \mathbb{R}^n]$; satisfy a local Lipschitz condition on the set $\mathbb{R}_+ \times S\rho$, $S\rho = [x \in \mathbb{R}^n : \|x\| \leq \rho < \infty]$ and $f(t,0) = 0$ for $t \geq 0$. A special case of (3.3) is where $F(t,y) = f(t,y) + R(t,y)$ and $R(t,y)$ is the perturbation term. Assume that the existence and uniqueness of the solutions $x(t) = x(t,t_0,x_0)$ of (3.1) for $t \geq t_0$ and $y(t) = y(t,\tau_0,y_0)$ of (3.3) for $t \geq t_0$.

The basic existence and uniqueness result with the Lipschitz condition by using contraction mapping theorem and a weighted norm with Mittag-Leffler function in [6, 10, 11, 12]. We introduce definitions for a variety of classes of functions that we use in Sections 4 and for generalized Dini-like derivatives and initial time difference strict stability in terms of two measures. All inequalities between vectors are componentwise.

Let us give the definition of the fractional strict stability in terms of two measures with initial time difference.

Definition 3.1: The solution $y(t,\tau_0,y_0)$ of the perturbed system (3.3) through (τ_0,y_0) is said to be initial time difference $(h_0 - h)-$strict stable in fractional case with respect to the solution $x(t-\eta,t_0,x_0)$, where $x(t,t_0,x_0)$ is any solution of the unperturbed system (3.1) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$. If given any $\epsilon_1 > 0$ and $\tau_0 \in \mathbb{R}_+$ there exist $\delta_1 = \delta_1(\epsilon_1,\tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon_1,\tau_0) > 0$ such that

$h(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) < \epsilon_1$ for $t \geq \tau_0$ whenever

$$ h(\tau_0, y_0 - x_0) < \delta_1 \text{ and } h_0(\tau_0, \tau_0 - t_0) < \delta_2 $$

and, for $\delta_1^* < \delta_1$ and $\delta_2^* < \delta_2$ there exist $0 < \epsilon_2 < \min\{\delta_1^*, \delta_2^*\}$ such that

$h(t, y(t,\tau_0,y_0) - x(t-\eta,t_0,x_0)) > \epsilon_2$ for $t \geq \tau_0$ whenever

$$ h(\tau_0, y_0 - x_0) > \delta_1^* \text{ and } h_0(\tau_0, \tau_0 - t_0) > \delta_2^*. $$

Definition 3.2: If δ_1, δ_2 and ϵ_2 in Definition 3.1 are independent of τ_0, then the solution $y(t,\tau_0,y_0)$ of the perturbed system (3.3) through (τ_0,y_0) is initial time difference $(h_0 - h)-$uniformly strict stable in fractional case with respect to the solution $x(t-\eta,t_0,x_0)$ for $t \geq \tau_0$.

Definition 3.3: The solution $y(t,\tau_0,y_0)$ of the system (3.3) through (τ_0,y_0) is said to be initial time difference

$(h_0-h)-$strictly attractive in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (3.1) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$. If given any $\alpha_1 > 0, \gamma_1 > 0, \epsilon_1 > 0$ and $\tau_0 \in \mathbb{R}_+$, for every $\alpha_2 < \alpha_1$ and $\gamma_2 < \gamma_1$, there exist $\epsilon_2 < \epsilon_1, T_1 = T_1(\epsilon_1, \tau_0)$ and $T_2 = T_2(\epsilon_1, \tau_0)$ such that

$$h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1,\ T_1 + \tau_0 \leq t \leq T_2 + \tau_0$$

whenever $h(\tau_0, y_0 - x_0) < \alpha_1$ and $h_0(\tau_0, \tau_0 - t_0) < \gamma_1$

and

$$h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) > \epsilon_2,\ T_2 + \tau_0 \geq t \geq T_1 + \tau_0$$

whenever $h(\tau_0, y_0 - x_0) > \alpha_2$ and $h_0(\tau_0, \tau_0 - t_0) > \gamma_2$.

If $T_1$and T_2 in Definition 3.3 are independent of τ_0, then the solution $y(t, \tau_0, y_0)$ of the system (3.3) is initial time difference $(h_0-h)-$strictly uniformly attractive in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ for $t \geq \tau_0$.

Definition 3.4: The solution $y(t, \tau_0, y_0)$ of the system (3.3) through (τ_0, y_0) is said to be initial time difference $(h_0-h)-$strictly asymptotically stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ if Definition 3.3 satisfies and the solution $y(t, \tau_0, y_0)$ of the perturbed system (3.3) through (τ_0, y_0) is initial time difference $(h_0 - h)-$strictly stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ of the unperturbed system (3.1).

If $T_1$and T_2 in Definition 3.3 are independent of τ_0, then the solution $y(t, \tau_0, y_0)$ of the system (3.3) is initial time difference $(h_0-h)-$uniformly strictly asymptotically stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ for $t \geq \tau_0$.

Definition 3.5: For any real-valued function $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, we define the fractional order Dini derivatives in Caputo's sense

$${}^cD^q_+V(t,x) = \lim_{h\to 0^+}\sup \frac{1}{h^q}[V(t,x) - V(t-h, x-h^q f(t,x))]$$

and

$${}^cD^q_-V(t,x) = \lim_{h\to 0^-}\inf \frac{1}{h^q}[V(t,x) - V(t-h, x-h^q f(t,x))]$$

where $x(t) = x(t, t_0, x_0)$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 3.6: For a real-valued function $V(t,x) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ we define the generalized fractional order derivatives (Dini-like derivatives) in Caputo's sense ${}^c_*D^q_+V(t, y-\widetilde{x})$ and ${}^c_*D^q_-V(t, y-\widetilde{x})$ as follows

$$\begin{aligned}&{}^c_*D^q_+V(t, y-\widetilde{x})\\ &= \lim_{h\to 0^+}\sup[\frac{V(t, y-\widetilde{x}) - V(t-h, y-\widetilde{x}-h^q H(t, y, \widetilde{x}))}{h^q}]\end{aligned}$$

$$\begin{aligned}&{}^c_*D^q_-V(t, y-\widetilde{x})\\ &= \lim_{h\to 0^-}\inf[\frac{V(t, y-\widetilde{x}) - V(t-h, y-\widetilde{x}-h^q H(t, y, \widetilde{x}))}{h^q}]\end{aligned}$$

where $H(t, y, \widetilde{x}) = F(t,y) - \widetilde{f}(t, \widetilde{x}))$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 3.7: The class $\mathcal{K}$ is set of functions such that $\mathcal{K} := [a : a \in C[[0,\rho], \mathbb{R}_+], a$ is strictly increasing and $a(0) = 0$ and also $a(t) \to \infty$ as $t \to \infty]$.

Definition 3.8: A function $h(t,x)$ is said to belong to the class Γ if $h \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, $\inf_{(t,x)} h(t,x) = 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 3.9: A function $h(t,x)$ is said to belong to the class Γ_0 if $h \in \Gamma$, $\sup_{t\in\mathbb{R}_+} h(t,x)$ exist for $x \in \mathbb{R}^n$.

4. MAIN RESULTS

In this section we obtain the strict stability concepts in fractional case with initial time difference parallel to the Lyapunov's results.

Theorem 4.1: Assume that

(A_1) for each $\mu, 0 < \mu < \rho, V_\mu \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_μ is locally Lipschitzian in z and for $(t,z) \in \mathbb{R}_+ \times S_\rho$ and $h(t,z) \geq \mu$,

$$b_1(h(t,z)) \leq V_\mu(t,z) \leq a_1(h(t,z)), a_1, b_1 \in \mathcal{K}$$

$${}^c_*D^q_+V_\mu(t,z) \leq 0; \tag{4.1}$$

(A_2) for each $\theta, 0 < \theta < \rho, V_\theta \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_θ is locally Lipschitzian in z and for $(t,z) \in \mathbb{R}_+ \times S_\rho$ and $h(t,z) \leq \theta$,

$$b_2(h(t,z)) \leq V_\theta(t,z) \leq a_2(h(t,z)), a_2, b_2 \in \mathcal{K}$$

$${}^c_*D^q_+V_\theta(t,z) \geq 0; \tag{4.2}$$

where $z(t) = y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)$ for $t \geq \tau_0$, $y(t, \tau_0, y_0)$ the solution of the system (3.3) through (τ_0, y_0), $x(t, t_0, x_0)$ is any solution of the system (3.1) for $t \geq \tau_0 \geq t_0 > 0$, and $\eta = \tau_0 - t_0$.

Then the solution $y(t, \tau_0, y_0)$ of the perturbed system (3.3) is the initial time difference $(h_0, h)-$strictly stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ of the unperturbed system, where $x(t, t_0, x_0)$ is any solution of the system (3.1) for $t \geq \tau_0 \geq t_0 > 0$.

Proof: Let $0 < \epsilon_1 < \rho$ and $\tau_0 \in \mathbb{R}_+$ and choose $\delta_1 = \delta_1(\epsilon_1, \tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon_1, \tau_0) > 0$ such that

$$a_1(\delta_1) < b_1(\epsilon_1) \tag{4.3}$$

since we have $b_1(\epsilon_1) \leq a_1(\delta_1)$ in (A_1). Then we claim that

$$h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1 \text{ for } t \geq \tau_0 \tag{4.4}$$

whenever $h(\tau_0, y_0 - x_0) < \delta_1$ and $h_0(\tau_0, \tau_0 - t_0) < \delta_2$.

If (4.4) is not true, then there would exist $t_1 > t_2 > \tau_0$ and the solution of (3.1) and from (4.1) with $h(\tau_0, y_0 - x_0) < \delta_1, h_0(\tau_0, \tau_0 - t_0) < \delta_2$ satisfying

$$h(t_1, y(t_1) - \widetilde{x}(t_1)) = \epsilon_1, h(t_2, y(t_2) - \widetilde{x}(t_2)) = \delta_1$$

and $\delta_1 \le h(t, y(t) - \widetilde{x}(t)) \le \epsilon_1$ for $t \in [t_2, t_1]$.

where $\widetilde{x}(t) = x(t-\eta, t_0, x_0)$.

Let us set $\mu = \delta_1$, and using (A_1) we get

$$\begin{aligned} b_1(\epsilon_1) &= b_1(h(t_1, y(t_1) - \widetilde{x}(t_1))) \\ &\le V_\mu(t_1, y(t_1) - \widetilde{x}(t_1)) \\ &\le V_\mu(t_2, y(t_2) - \widetilde{x}(t_2)) \\ &\le a_1(h(t_2, y(t_2) - \widetilde{x}(t_2))) \\ &= a_1(\delta_1) \end{aligned}$$

and we have the inequality

$$b_1(\epsilon_1) \le a_1(\delta_1)$$

which contradicts with (4.3). Hence (4.4) is valid.

Now let $0 < \delta_1^* < \delta_1, 0 < \delta_2^* < \delta_2$ and choose $0 < h(\tau_0, y_0 - x_0) < \delta_1^* < \delta_1$ and $0 < h_0(\tau_0, \tau_0 - t_0) < \delta_2^* < \delta_2$ for $0 < \epsilon_2 < \delta = \min\{\delta_1^*, \delta_2^*\}$ such that

$$a_2(\epsilon_2) < b_2(\delta). \tag{4.5}$$

Then we can prove that

$$\epsilon_2 < h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1 \text{ for } t \ge \tau_0 \tag{4.6}$$

whenever $\delta_1^* < h(\tau_0, y_0 - x_0) < \delta_1$ and $\delta_2^* < h_0(\tau_0, \tau_0 - t_0) < \delta_2$.

If (4.6) is not true, then there would exist $t_1 > t_2 > \tau_0$ and the solution of (3.1) and (4.2) with $\delta_1^* < h(\tau_0, y_0 - x_0) < \delta_1, \delta_2^* < h_0(\tau_0, \tau_0 - t_0) < \delta_2$ satisfying

$$\begin{gathered} h(t_1, y(t_1) - \widetilde{x}(t_1)) = \epsilon_2, \\ h(t_2, y(t_2) - \widetilde{x}(t_2)) = \delta \\ \text{and } h(t, y(t) - \widetilde{x}(t)) \le \delta \text{ for } t \in [t_2, t_1]. \end{gathered} \tag{4.7}$$

Let us set $\theta = \delta$ and using (A_2), we get

$$\begin{aligned} a_2(\epsilon_2) &= a_2(h(t_1, y(t_1) - \widetilde{x}(t_1))) \\ &\ge V_\theta(t_1, y(t_1) - \widetilde{x}(t_1)) \\ &\ge V_\theta(t_2, y(t_2) - \widetilde{x}(t_2)) \\ &\ge b_2(h(t_2, y(t_2) - \widetilde{x}(t_2))) \\ &= b_2(\delta) \end{aligned}$$

and we have the inequality

$$a_2(\epsilon_2) \ge b_2(\delta)$$

which contradicts with (4.5). Thus (4.6) is valid.

Then the solution $y(t, \tau_0, y_0)$ of the perturbed system (3.3) through (τ_0, y_0) is initial time difference $(h_0, h)-$strictly stable in fractional case with respect to the solution of unperturbed system $x(t-\eta, t_0, x_0)$ for $t \ge \tau_0$.

This completes the proof of Theorem 4.1. □

If δ_1, δ_2 and ϵ_2 in the proof of the Theorem 4.1 are chosen independent of τ_0, then the solution $y(t, \tau_0, y_0)$ of the perturbed system (3.3) is initial time difference $(h_0 - h)-$strictly uniformly stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ for $t \ge \tau_0$.

Theorem 4.2: Assume that

(A_1) for each $\mu, 0 < \mu < \rho, V_\mu \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_μ is locally Lipschitzian in z and for $(t, z) \in \mathbb{R}_+ \times S_\rho$ and $h(t, z) \ge \mu$,

$$b_1(h(t, z)) \le V_\mu(t, z) \le a_1(h(t, z)), a_1, b_1 \in \mathcal{K},$$

$${}_*^cD_+^q V_\mu(t, z) \le -c_1(h(t, z)) \;, c_1 \in \mathcal{K}; \tag{4.8}$$

(A_2) for each $\theta, 0 < \theta < \rho, V_\theta \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_θ is locally Lipschitzian in z and for $(t, z) \in \mathbb{R}_+ \times S_\rho$ and $h(t, z) \le \theta$,

$$b_2(h(t, z)) \le V_\theta(t, z) \le a_2(h(t, z)), a_2, b_2 \in \mathcal{K},$$

$${}_*^cD_+^q V_\theta(t, z) \ge -c_2(h(t, z)) \; c_2 \in \mathcal{K}; \tag{4.9}$$

where $z(t) = y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)$ for $t \ge \tau_0, y(t, \tau_0, y_0)$ of the perturbed system (3.3) through (τ_0, y_0) and $x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the unperturbed system (3.1) for $t \ge \tau_0 \ge t_0 > 0$.

Then the solution $y(t, \tau_0, y_0)$ of the perturbed system (3.3) through (τ_0, y_0) is the initial time difference $(h_0, h)-$strictly uniformly asymptotically stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ of the unperturbed system, where $x(t, t_0, x_0)$ is any solution of the unperturbed system (3.1) for $t \ge \tau_0 \ge t_0 > 0$.

Proof: We note that (4.8) implies (4.1). However, (4.9) does not yield (4.2). Therefore, we get because of (4.8) only $(h_0, h)-$uniformly stability in fractional case of perturbed systems related to initial time difference with respect to unperturbed systems that is for given any $\epsilon_1 \le \rho$ and $\tau_0 \in \mathbb{R}_+$ there exist $\delta_{10} = \delta_{10}(\epsilon_1) > 0$ and $\delta_{20} = \delta_{20}(\epsilon_1) > 0$ such that

$$\begin{gathered} h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1 \text{ for } t \ge \tau_0 \text{ whenever} \\ h(\tau_0, y_0 - x_0) < \delta_{10} \text{ and } h_0(\tau_0, \tau_0 - t_0) < \delta_{20}. \end{gathered} \tag{4.10}$$

To prove the conclusion of Theorem 4.2 we need to show that the solution $y(t, \tau_0, y_0)$ of the system (3.3) through (τ_0, y_0) is initial time difference $(h_0, h)-$strictly uniformly attractive in fractional case with respect to $x(t-\eta, t_0, x_0)$ for this purpose, let $\epsilon_1 = \rho$ and set $\delta_{10} = \delta_1(\rho)$ and $\delta_{20} = \delta_2(\rho)$ so that (4.10) yields

$$h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \rho \text{ for } t \ge \tau_0 \text{ whenever}$$

$$h(\tau_0, y_0 - x_0) < \delta_{10} \text{ and } h_0(\tau_0, \tau_0 - t_0) < \delta_{20}.$$

Let $h(\tau_0, y_0 - x_0) < \delta_{10}$ and $h_0(\tau_0, \tau_0 - t_0) < \delta_{20}$. We show, using standard argument, that there exists a $t^* \in [\tau_0, \tau_0 + T]$, we choose $T = T(\epsilon, \tau_0) \ge \left(\frac{a_1(\max\{\delta_{10}, \delta_{20}\})}{c_1(\min\{\delta_1, \delta_2\})}\Gamma(q+1)\right)^{\frac{1}{q}}$ where δ_{10} and δ_{20} are the numbers corresponding to ϵ_1 in (4.10) that is in initial time difference $(h_0, h)-$uniformly stability in fractional case of perturbed system with respect to $x(t-\eta, t_0, x_0)$ such that $h(\tau_0, y(t^*, \tau_0, y_0) - x(t^* - \eta, t_0, x_0)) < \delta_1$ for any solutions of the system (3.3) with

$h(\tau_0, y_0 - x_0) < \delta_{10}$ and $h_0(\tau_0, \tau_0 - t_0) < \delta_{20}$. If this is not true, we will have $h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) \geq \delta_1$ for $t \in [\tau_0, \tau_0 + T]$. Then, $\mu = \delta_1$ and using (A_1) with (4.8), we have

$$\begin{aligned}
0 < b_1(\delta_1) &\leq b_1(h(\tau_0 + T, y(\tau_0 + T) - \widetilde{x}(\tau_0 + T))) \\
&\leq V_\mu(\tau_0 + T, y(\tau_0 + T) - \widetilde{x}(\tau_0 + T)) \\
&\leq V_\mu(\tau_0, y_0 - x_0) \\
&\quad - \frac{1}{\Gamma(q)} \int_{\tau_0}^{\tau_0+T} (t-s)^{q-1} c_1(h(s, y(s) - \widetilde{x}(s)))ds \\
&\leq a_1(\max\{\delta_{10}, \delta_{20}\}) \\
&\quad - \frac{c_1(\min\{\delta_1, \delta_2\})}{\Gamma(q)} \int_{\tau_0}^{\tau_0+T} (t-s)^{q-1} ds \\
&\leq a_1(\max\{\delta_{10}, \delta_{20}\}) - \frac{c_1(\min\{\delta_1, \delta_2\})}{\Gamma(q+1)} T^q \\
&\leq 0
\end{aligned}$$

in view of the choice of T. This contradiction implies that there exist a $t^* \in [\tau_0, \tau_0 + T]$ satisfying $h(\tau_0, y(t^*, \tau_0, y_0) - x(t^* - \eta, t_0, x_0)) < \delta_1$. Due to the $(h_0, h)-$uniform stability in fractional case $y(t, \tau_0, y_0)$ of the unperturbed systems with initial time difference with respect to $x(t-\eta, t_0, x_0)$, this yields that

$$h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1 \text{ for } t \geq \tau_0 + T \geq t^*$$

which implies that there exists a $\tau_0 < T_1 < T$ such that

$$h(\tau_0 + T, y(\tau_0 + T) - x(\tau_0 + T - \eta)) = \epsilon_1$$

Now, for any $\delta_{12}, 0 < \delta_{12} < \delta_{10}$ and $0 < \delta_{12} < \delta_{20}$ we choose ϵ_2 such that $b_2(\delta_{12}) > a_2(\epsilon_2)$ and $0 < \epsilon_2 < \epsilon_1 < \delta_{12}$.

Suppose that $\delta_{12} < h(\tau_0, y_0 - x_0) < \min\{\delta_{10}, \delta_{20}\}$ and $\delta_{12} < h_0(\tau_0, \tau_0 - t_0) < \min\{\delta_{10}, \delta_{20}\}$.Let us define $\tau = [\frac{\Gamma(q)(b_2(\epsilon_1) - a_2(\epsilon_2))}{c_2(\epsilon_1)}]^{\frac{1}{q}}$, and $T_2 = T_1 + \tau$.Since $h(t, y(t) - \widetilde{x}(t)) \leq \epsilon_1$ for $t \geq \tau_0 + T_1$, choosing $\theta = \epsilon_1$ and using (A_2) with (4.9) we have for $t \in [\tau_0 + T_1, \tau_0 + T_2]$,

$$a_2(\left\|y(t) - \widetilde{x}(t)\right\|) \geq V_\theta(t, y(t) - \widetilde{x}(t))$$

$$\begin{aligned}
&\geq V_\theta(\tau_0 + T_1, y(\tau_0 + T_1) - \widetilde{x}(\tau_0 + T_1)) \\
&\quad - \frac{1}{\Gamma(q)} \int_{\tau_0+T_1}^{t} (t-s)^{q-1} c_2(h(s, y(s) - \widetilde{x}(s)))ds \\
&\geq b_2(\epsilon_1) \\
&\quad - \frac{1}{\Gamma(q)} \int_{\tau_0+T_1}^{t} (t-s)^{q-1} c_2(h(s, y(s) - \widetilde{x}(s)))ds \\
&\geq b_2(\epsilon_1) - \frac{c_2(\epsilon_1)}{\Gamma(q)} [t - (\tau_0 + T_1)]^q.
\end{aligned}$$

Since, $t - (\tau_0 + T_1) > \tau$ and a_2^{-1} exists, it follows that

$$\begin{aligned}
a_2(\left\|y(t) - \widetilde{x}(t)\right\|) &> b_2(\epsilon_1) - \frac{c_2(\epsilon_1)}{\Gamma(q)} [\frac{\Gamma(q)(b_2(\epsilon_1) - a_2(\epsilon_2))}{c_2(\epsilon_1)}] \\
&= a_2(\epsilon_2).
\end{aligned}$$

This yields that

$$h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) \geq \epsilon_2 \text{ for } t \in [\tau_0 + T_1, \tau_0 + T_2]$$

and therefore,

$$\epsilon_2 < h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1, t \in [\tau_0 + T_1, \tau_0 + T_2].$$

Then the solution $y(t, \tau_0, y_0)$ of the perturbed system (3.3) through (τ_0, y_0) is initial time difference $(h_0, h)-$ strictly uniformly asymptotically stable in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the unperturbed system (3.1) for $t \geq \tau_0 \geq t_0 > 0$.This completes the proof. □

Before we express the comparison result in fractional case, we need to give uncoupled comparison fractional order differential systems and to define $(h_0, h)-$strictly stability in fractional case of comparison fractional order differential systems. Consider the uncoupled comparison fractional order differential systems:

$$\begin{cases} (i)^c D^q\, u_1 = g_1(t, u_1), u_1(\tau_0) = u_{10} \geq 0 \\ (ii)^c D^q\, u_2 = g_2(t, u_2), u_2(\tau_0) = u_{20} \geq 0 \end{cases} \tag{4.11}$$

where $g_1, g_2 \in C\left[\mathbb{R}_+^2, \mathbb{R}\right]$. The fractional order comparison system (4.11) is said to be $(h_0, h)-$strictly stable in fractional case:

If given any $\epsilon_1 > 0$ and $t \geq \tau_0, \tau_0 \in \mathbb{R}_+$, there exist a $\delta_1 > 0$ such that

$$h_0(\tau_0, u_{10}) < \delta_1 \text{ implies } h(t, u_1(t)) < \epsilon_1 \text{ for } t \geq \tau_0$$

and for every $\delta_2 \leq \delta_1$ there exists an $\epsilon_2, 0 < \epsilon_2 < \delta_2$ such that

$$h_0(\tau_0, u_{20}) > \delta_2 \text{ implies } h(t, u_2(t)) > \epsilon_2 \text{ for } t \geq \tau_0.$$

Here, $u_1(t)$ and $u_2(t)$ are any solutions of (i) in (4.11) and (ii) in (4.11); respectively. Following this theorem based on this definition and that theorem is formulated in terms of comparison principle. □

Theorem 4.3: Assume that

(A_1) for each $\mu, 0 < \mu < \rho, V_\mu \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_μ is locally Lipschitzian in z and for $(t, z) \in \mathbb{R}_+ \times S_\rho$ and $h(t, z) \geq \eta$,

$$b_1(h(t, z)) \leq V_\mu(t, z) \leq a_1(h(t, z)), a_1, b_1 \in \mathcal{K},$$

$${}_*^c D_+^q V_\mu(t, z) \leq g_1(t, V_\mu(t, z)); \tag{4.12}$$

(A_2) for each $\theta, 0 < \theta < \rho, V_\theta \in C[\mathbb{R}_+ \times S_\rho, \mathbb{R}_+]$ and V_θ is locally Lipschitzian in z and for $(t, z) \in \mathbb{R}_+ \times S_\rho$ and $h(t, z) \leq \theta$,

$$b_2(h(t, z)) \leq V_\theta(t, z) \leq a_2(h(t, z)), a_2, b_2 \in \mathcal{K},$$

$${}_*^c D_+^q V_\theta(t, z) \geq g_2(t, V_\theta(t, z)); \tag{4.13}$$

where $g_2(t, u) \leq g_1(t, u), g_1, g_2 \in C[\mathbb{R}_+^2, \mathbb{R}], g_1(t, 0) \equiv g_2(t, 0) \equiv 0$ and $z(t) = y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)$ for $t \geq \tau_0, y(t, \tau_0, y_0)$ of the system (3.3) through (τ_0, y_0) and

$x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (3.1) for $t \geq \tau_0 \geq t_0 > 0$.

Then any $(h_0, h)-$strict stability concept in fractional case of the comparison system implies the corresponding $(h_0, h)-$strict stability concept in fractional case of the solution $y(t, \tau_0, y_0)$ of the perturbed system (3.3) through (τ_0, y_0) with respect to the solution $x(t-\eta, t_0, x_0)$ of the unperturbed system (3.1) with initial time difference where $x(t, t_0, x_0)$ is any solution of the unperturbed system (3.1) for $t \geq \tau_0 \geq t_0 > 0$.

Proof: First we will prove the case of initial time difference $(h_0, h)-$strictly uniformly stability in fractional case of the perturbed system with respect to the unperturbed system. Suppose that the comparison differential systems in (4.11) is $(h_0, h)-$strictly uniformly stable in fractional case, then for any given $\epsilon_1,\ 0 < \epsilon_1 < \delta$, there exist a $\delta^* > 0$ such that $0 < u_{10} < \delta^*$ implies that $u_1(t, \tau_0, u_{10}) < b_1(\epsilon_1)$ for $t \geq \tau_0$ (4.14)

where $u_1(t) = u_1(t, \tau_0, u_{10})$ is the solution of (4.11).

For this $\epsilon_1 > 0$, we choose $\delta_1 > 0$ and $\delta_{11} > 0$, such that $a_1(\delta_1) \leq \delta^*$ and $\delta_1^* < \epsilon_1$ where $\delta_1^* = \max\{\delta_1, \delta_{11}\}$, then we claim that

$$h(\tau_0, y_0 - x_0) < \delta_1, h_0(\tau_0, \tau_0 - t_0) < \delta_{11} \Rightarrow \tag{4.15}$$

$$h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1, t \geq \tau_0.$$

If it is not true, then there exist t_1 and $t_2,\ t_2 > t_1 > \tau_0$ and a solution of

$${}^cD^q z = \tilde{f}(t, z),\ z(\tau_0) = y_0 - x_0 \text{ for } t \geq \tau_0 \text{ with}$$

$h_0(\tau_0, \tau_0 - t_0) < \delta_{11}$ and $h(\tau_0, y_0 - x_0) < \delta_1$.

$$\begin{aligned} &h(t_1, y(t_1, \tau_0, y_0) - x(t_1 - \eta, t_0, x_0)) = \delta_1^*,\\ &h(t_2, y(t_2, \tau_0, y_0) - x(t_2 - \eta, t_0, x_0)) = \epsilon_1 \text{ and} \end{aligned}$$

$$\delta_1^* < h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_1 \text{ for } [t_1, t_2].$$

Choosing $\mu = \delta_1^*$ and using the theory of differential inequalities, together with (A_1), we obtain (4.12) and (4.14)

$$\begin{aligned} b_1(\epsilon_1) &= b_1(h(t_2, y(t_2, \tau_0, y_0) - x(t_2 - \eta, t_0, x_0)))\\ &\leq V_\mu(t_2, y(t_2, \tau_0, y_0) - x(t_2 - \eta, t_0, x_0))\\ &\leq r(t_2, t_1, V_\mu(t_1, y(t_1, \tau_0, y_0) - x(t_1 - \eta, t_0, x_0)))\\ &\leq r(t_2, t_1, a_1(\delta_1))\\ &\leq r(t_2, t_1, \delta^*)\\ &< b_1(\epsilon_1) \end{aligned}$$

which is a contradiction. Here $r(t, t_0, u_{10})$ is the maximal solution of (4.11). Hence, (4.15) is true and we have initial time difference $(h_0, h)-$strictly uniformly stability in fractional case.

Now, we shall prove initial time difference $(h_0, h)-$strictly uniformly attractive in fractional case.

For any given $\delta_2,\ \epsilon_2 > 0,\ \delta_2 < \delta^*$ we choose $\bar{\delta}_2$and $\bar{\epsilon}_2$such that $a_1(\delta_2) < \bar{\delta}_2$and $b_1(\epsilon_2) \geq \bar{\epsilon}$. For these $\bar{\delta}_2$ and $\bar{\epsilon}_2$, since (4.11) is strictly uniformly attractive in fractional case, for any $\bar{\delta}_3 < \bar{\delta}_2$there exist $\bar{\epsilon}_3$ and T_1 and T_2 (we assume $T_2 < T_1$) such that $\bar{\delta}_3 < u_{10} = u_{20} < \bar{\delta}_2$ implies

$$r(t, \tau_0, u_{10}) \leq r(t, \tau_0, \bar{\delta}_2) < \bar{\epsilon}_2$$

$$\rho(t, \tau_0, u_{20}) \geq \rho(t, \tau_0, \bar{\delta}_3) > \bar{\epsilon}_2$$

where $r(t, \tau_0, u_{10})$ and $\rho(t, \tau_0, u_{20})$ is the maximal solution and minimal solution of (4.11) (i) and (4.11) (ii); respectively.

Now, for any δ_3, let $b_2(\delta_3) \geq \bar{\delta}_3$. We choose ϵ_3 such that $a_2(\epsilon_3) < \bar{\epsilon}_3$. Then by using comparison principle in fractional case (4.11), (i) and (A_1), we have

$$\begin{aligned} &b_1(h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)))\\ &\leq V_\mu(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)))\\ &\leq r(t, \tau_0, V_\mu(\tau_0, y_0 - x_0))\\ &\leq r(t, \tau_0, a_1(h(\tau_0, y_0 - x_0)))\\ &\leq r(t, \tau_0, \bar{\delta}_2)\\ &< \bar{\epsilon}_2 \leq b_1(\epsilon_2) \end{aligned}$$

$$b_1(h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0))) < b_1(\epsilon_2) \tag{4.16}$$

which implies that $h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_2$ for $t \in [\tau_0 + T_2, \tau_0 + T_1]$.

Similarly, by using comparison principle in fractional case (4.11), (ii) and (A_2), we get

$$\begin{aligned} &a_2(h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)))\\ &\geq V_\theta(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0))\\ &\geq \rho(t, \tau_0, V_\theta(\tau_0, y_0 - x_0))\\ &\geq \rho(t, \tau_0, b_2(h(\tau_0, y_0 - x_0))\\ &\geq \rho(t, \tau_0, b_2(\delta_3))\\ &\geq \rho(t, \tau_0, \bar{\delta}_3)\\ &> \bar{\epsilon}_3 \geq a_2(\epsilon_3) \end{aligned}$$

$$a_2(h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0))) > a_2(\epsilon_3) \tag{4.17}$$

which implies that for $h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) > \epsilon_3$ for $\tau_0 + T_2 < t < \tau_0 + T_1$. Thus (4.16) and (4.17) yield that $\epsilon_3 < h(t, y(t, \tau_0, y_0) - x(t-\eta, t_0, x_0)) < \epsilon_2,\ \tau_0 + T_2 < t < \tau_0 + T_1$ provided that $\delta_2 < h(\tau_0, y_0 - x_0) < \delta_1$ and $\delta^* < h_0(\tau_0, \tau_0 - t_0) < \delta_{11}$. Hence, the solution $y(t, \tau_0, y_0)$ of the perturbed system of (3.3) through (τ_0, y_0) is initial time difference $(h_0, h)-$strictly uniformly attractive in fractional case with respect to the solution $x(t-\eta, t_0, x_0)$ of the unperturbed system where $x(t, t_0, x_0)$ is any solution of the unperturbed system of (3.1) for $t \geq \tau_0 \geq t_0 > 0$. Therefore the proof is completed.

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