FREE NONLINEAR NONCLASSICAL OSCILLATIONS OF MICROSCALE BEAMS

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Abstract: Free vibration analysis of microscale beams is investigated in this paper. The nonlinear model is conducted within the context of non-classical continuum mechanics, by introducing a material length scale parameter. The nonlinear equation of motion is derived by using a combination of the modified couple stress theory and Hamilton's principle. Based on this newly developed model, calculations have been performed for microbeams simply supported between two immobile supports. The nonlinear frequencies of a beam with initial lateral displacement are discussed. It is shown that the size effect is significant when the ratio of characteristic thickness to internal material length scale parameter is approximately equal to one, but is diminishing with the increase of the ratio. Our results also indicate that the nonlinearity has a great effect on the vibration behavior of microscale beams. To attain accurate and reliable characterization of the vibration properties of microscale beams, therefore, both the microstructure- dependent parameters and the nonlinearities have to be incorporated in the design of microscale beam devices and systems.

Keywords: Microscale beam, Free vibration, Non-classical beam model, nonlinear

1. INTRODUCTION

Miniaturized beams are the core structures widely used in sensors, actuators, microscopes, MEMS and NEMS [1–4] for applications ranging from sensing and communications to energy harvesting, fundamental studies of quantum mechanical systems, etc. Across these applications, the characteristic thicknesses of the beams are typically on the order of microns or even sub-microns. As reported in many papers (e.g., [5–8]), the microscale beams may be made of metals, polymer, traditional silicon-based materials or functionally graded materials (FGMs). The design of microbeams is dominated by several basic requirements. One of these basic requirements is to attain mechanical and vibration properties to match the required functionality of interest. It is not surprising, therefore, that the literature on this topic is constantly expanding.

In the past decades, many theoretical studies of microscale beams were based on the classical continuum theory (see, e.g., [9–13]). In the current work, however, theoretical analysis will not be conducted within the context of classical continuum mechanics. Though the classical continuum models are relevant to some extent, the length scales associated with material's microstructure (such as lattice spacing between individual atoms) are often sufficiently small to call the applicability of classical continuum models into question [14]. Indeed, the size dependence of material deformation behavior in micronscale has been observed This modified couple stress theory has also been used to study the dynamic properties (e.g., natural frequencies) of experimentally in the last two decades. Related work on this topic appears to have started in the 1990s. Some of the key contributions in this area were made by Fleck et al. [15], Ma and Clarke [16], Stolken and Evans [17], Chong and Lam [18], Lam et al. [19], and McFarland and Colton [20]. The size dependence phenomenon has been observed in the materials of either metals or polymers. In these experimental works, the microscale structures studied may be copper wires, silver single crystal, nickel beams, or epoxy polymeric beams. These experimental results certainly demonstrate that the size dependence is intrinsic to certain materials with microstructures.

Since beam models based on classical elasticity theory are not capable of describing the size effects, the nonlocal, strain gradient and classical couple stress elasticity theories were used to develop the size-dependent beam models (see, e.g., [7,8,13,14,21–33]). Recently, Yang et al. [34] proposed a modified couple stress theory, in which the constitutive equations contain only one single additional internal material length scale parameter besides two classical material constants. Owing its advantageous expression, the modified couple stress (non-classical) theory has attracted many researchers in the past years. As an example, Park and Gao [7] have studied the statically mechanical properties of Bernoulli–Euler cantilever beams by using this non-classical elasticity theory. The corresponding results were applied to explain bending test of epoxy polymeric beams successfully.

Euler–Bernoulli microbeams by Kong et al. [35]. In their study, two boundary value problems (one for simply supported beam and another for cantilevered beam) were solved and size effect on the natural frequencies for these two kinds of boundary conditions were evaluated. It was found that the natural frequencies of the mircobeams predicted by the modified couple stress theory are generally higher than those predicted by the classical Euler–Bernoulli beam theory. More recently, Ma et al. [23] further developed a microstructure-dependent Timoshenko beam model by using the modified couple stress theory. In their model, both bending and axial deformations were considered, and the Poisson effect was incorporated. Of course, this refined Timoshenko beam model can be easily reduced to the classical Timoshenko beam model.

From the literature mentioned above, it can be found that, although, several microstructure-dependent models have been initiated to discuss the static and dynamic properties of microscale beams in the past years, these studies were still based on linear theories, i.e., nonlinearities in the microbeam systems were excluded absolutely. As reported in a recent paper by Kong et al. [35], accurate characterization of the static and dynamic properties of microstructures is urgent and vital for reliable and optimal design of MEMS devices. Although the linear theories are relevant to some extent, the effect of nonlinearities on the static and dynamical behaviors of microbeams may be pronounced and has to be considered in many situations (e.g., to attain accurate natural frequency of vibration in order to enable sensing and to match the frequencies of the signals of interest). Another aspect is that the linear theories are not capable of predicting the static configurations when the microbeam undergoes a buckling instability. This motivates the work presented in this paper.

The objective of the present paper is to establish a nonlinear non-classical Euler–Bernoulli beam model for microscale beams by using the modified couple stress theory. The beam material is assumed to obey the modified couple stress theory, as developed by Yang et al. [34]. This new nonlinear model contains a material length scale parameter and can capture the size effect. The nonlinear equation of motion will be derived by using the Hamilton's principle. The nonlinear term added, assumed supported between two axially immobile supports. Based on the equation of motion derived, the free vibration of pinned–pinned microbeams will be studied. It will be shown that the effect of material length scale parameter and nonlinearity on the vibration frequencies are significant. The difference between the nonlinear non-classical results and the linear results (both classical and non-classical) will be quantitatively shown and analyzed.

2. Formulation

The system under consideration is a microscale beam of length L between two immovable supports, mass density ρ, cross-section height h and cross-section width b. The cross-section of the beam is symmetric (either rectangular or circular). We will consider the nonlinear vibrations of microbeams with transverse dimensions ranging from several micro-meters to hundreds of micro-meters.

It will be useful, later on, to have handy the linear problem of a microscale beam. The linear equation of motion based on a modified couple stress elasticity theory is given by [35]

$$\left(EI+GA\mathbf{l}^2\right)\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial t^2}=q(x,t) \tag{1}$$

where E is the Young's modulus of elasticity, I is the moment of inertia of the cross-section, A is the cross-sectional area, G is a Lame's constant ($G = E/[2(1 + \mu)]$ is also known as the shear modulus, where μ is Poisson's ratio), ℓ is a material length scale parameter, $q(x,t)$ is a transverse loading, and $w(x, t)$ is the lateral deflection of the beam; x and t are the axial coordinate and time, respectively.

Since the derivation of Eq. (1) is based on a refined Euler–Bernoulli beam theory, the corresponding theoretical model described is called ''non-classical Euler–Bernoulli beam model". It is immediately found that the above equation has introduced a material length scale parameter ℓ, which represents the microstructure-dependent effect.

As reported by Sadeghian et al. [36], the choice of appropriate structural analysis model of the microscale beam depends on the magnitude of the lateral deflection compared to the thickness of the beam. The theoretical model described in Eq. (1) may be an adequate representation for the case that the deflection is considerably small (e.g., the deflection is smaller than the thickness of the microbeam). For the case that the deflection is relatively large (e.g., the deflection is approximately equal to or larger than the thickness of the microbeam), bending-stretching coupling terms need to be taken into account, since the effects of nonlinearity on the mechanical and vibration properties become observable.

The nonlinear equation of motion of a microbeam with immovable ends will be formulated by using the Hamilton's principle. According to the modified couple stress theory [34], the bending strain energy U_m of the microbeam is a function of both the strain (conjugated with stress) and the curvature (conjugated with couple stress). Then the bending strain energy in a deformed microbeam is given by [7]

$$U_m=-\frac{1}{2}\int_0^L M_x\frac{\partial^2 w}{\partial x^2}dx-\frac{1}{2}\int_0^L g_{xy}\frac{\partial^2 w}{\partial x^2}dx \tag{2}$$

where the resultant moment M_x and the couple moment Y_{xy} are defined, respectively, by

$$M_x=\int_A s_{xx}z\,dA \tag{3}$$

$$Y_{xy}=\int_A m_{xy}\,dA \tag{4}$$

In the above two equations, σ_{xx} and m_{xy} are, respectively, defined by

$$s_{xx}=-Ez\frac{\partial^2 w}{\partial x^2} \tag{5}$$

$$m_{xy}=-G\mathbf{l}^2\frac{\partial^2 w}{\partial x^2} \tag{6}$$

Then U_m may be rewritten as

$$U_m = \frac{1}{2}\int_0^L \left(EI + GA\mathbf{l}^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \tag{7}$$

By neglecting the body force and body couple, the work done by the externally transverse loading $q(x, t)$ may be written as

$$W = \int_0^L q(x,t)w(x)dx \tag{8}$$

The kinetic energy of the microbeam is given by

$$K = \frac{1}{2}\int_0^L r\,A\left(\frac{\partial w}{\partial t}\right)^2 dx = \frac{m}{2}\int_0^L \left(\frac{\partial w}{\partial t}\right)^2 dx \tag{9}$$

in which $m = \rho A$ is the beam mass of per unit length.
According to the Hamilton's principle, the dynamic equation of motion of this beam as well as all possible boundary conditions can be derived by using the following variational equation

$$d\int_{t_1}^{t_2}\left(K - U_m - W\right)dt = 0 \tag{10}$$

Substituting Eqs. (7)–(9) into Eq. (10), one obtains

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_0^L \{-\left(EI + GA\mathbf{l}^2\right)\frac{\partial^4 w}{\partial x^4} - m\frac{\partial^2 w}{\partial t^2} \\
&+q - \int_{t_1}^{t_2}\left[\left(EI + GA\mathbf{l}^2\right)\frac{\partial^3 w}{\partial x^3}d\,w\right]_0^L dt \\
&-\int_{t_1}^{t_2}\left[\left(EI + GA\mathbf{l}^2\right)\frac{\partial^2 w}{\partial x^2}d\,w'\right]_0^L dt \\
&+\int_0^L \left[\frac{\partial w}{\partial t}d\,w\right]_{t_1}^{t_2} dx = 0
\end{aligned} \tag{11}$$

In view of Eq. (11), the nonlinear equation of motion of the beam in terms of w(x, t) is given by

$$\left(EI + GA\mathbf{l}^2\right)\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = q(x,t) \tag{12}$$

and the boundary conditions are

$$\frac{\partial^3 w}{\partial x^3} = 0 \text{ or } w = 0 \text{ at } x = 0 \text{ and } x = L, \tag{13a}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \text{ or } \frac{\partial w}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = L. \tag{13b}$$

It can be seen from Eq. (12) that the deflections of the beam are related to two type of material parameters: one associated with ρA, EA and EI as in classical beam model and the other associated with $GA\ell^2$. Therefore, the current refined Euler–Bernoulli beam model based on the modified couple stress elasticity theory contains one additional internal material constant besides three classical material parameters. As can be expected, the presence of ℓ enables us to analyze the size effect.
Defining the following quantities

$$x = x/L,\quad h = w/L,\quad t = \left[\frac{EI}{b\,mL^4}\right]^{1/2} t$$

$$a = L/r,\quad r = \sqrt{I/A},\quad b = \frac{1}{1+\left[6/(1+m)\right]\times(\mathbf{l}/h)^2}$$

$$\begin{aligned}
&\Gamma = \frac{T_0 L^2}{EI + GA\mathbf{l}^2} = b\,\frac{T_0 L^2}{EI} = b\Gamma_L, \\
&f = \frac{qL^3}{\left(EI + GA\mathbf{l}^2\right)} = b\,\frac{qL^3}{EI}, \\
&k = \frac{EAL^2}{2\left(EI + GA\mathbf{l}^2\right)} = \frac{b}{2}a^2
\end{aligned} \tag{14}$$

Eq. (12) may be written in the dimensionless form

$$\frac{\partial^4 h}{\partial x^4} + \left[\Gamma - k\int_0^1\left(\frac{\partial h}{\partial x}\right)^2 dx\right]\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial t^2} = f \tag{15}$$

Since Eq. (15) is represented in a dimensionless form, the current non-classical beam model may be used to analyze the dynamic responses of microbeams, regardless of the beam materials or length scales. It may be also mentioned that the equation of motion, (15), is essentially based on the Euler–Bernoulli beam assumptions. Like all other analytical models, therefore, the newly developed beam model has limitations, which are contingent upon the applicability of the modified couple stress theory. Specifically, the microbeam must be slender so that the Euler–Bernoulli beam assumptions are applicable. For microbeam with relatively large width (b), however, the current Euler–Bernoulli beam theory may be inadequate for predicting the response of microbeams.

In this paper, the microbeam under consideration is assumed to be pinned–pinned. For such a beam system, the deflection and moment are zero at both ends. Then the boundary conditions can be written in the dimensionless form

$$h = 0 \quad \text{and} \quad \frac{\partial^2 h}{\partial x^2} = 0 \quad \text{at} \quad x = 0,1 \tag{16}$$

Before closing this section, it should be mentioned that, for analysis convenience, the beam material is chosen to be epoxy. Thus, the material constants used here are E = 1.44 GPa and $\mathbf{l} = 1.76\ m\text{m}$ [7]. In the following analysis, for comparison purpose, we will choose a = 30 and two different values of Poisson's ratio (i.e., $\mu = 0$ and $\mu = 0.38$).

3. Free vibration

In this section, the free vibration of a microscale beam with both ends immoveable will be analyzed. It is assumed that the external transverse force $q(x,t)$ is absent. Based on this assumption, Eq. (15) becomes

$$\frac{\partial^4 h}{\partial x^4} - \left[k\int_0^L\left(\frac{\partial h}{\partial x}\right)^2 dx\right]\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial t^2} = 0 \tag{17}$$

As already mentioned in the foregoing, the nonlinearity is caused by the immoveable ends which are not allowed to move to any appreciable extent relative to the initial coordinates of the beam ends. Therefore, the axial inertia may be also neglected.
The initial conditions considered in the current work are:

$$w\left(L/2,0\right)=w_{\max}, \quad \frac{\partial w\left(L/2,0\right)}{\partial t}=0 \tag{18}$$

The dimensionless initial conditions given by Eq. (18) become

$$\eta\left(1/2,0\right)=w_{\max}, \quad \frac{\partial w\left(1/2,0\right)}{\partial t}=0 \tag{19}$$

Assume that

$$\eta\left(x,t\right)=\psi\left(x\right)q\left(t\right) \tag{20}$$

where $\psi(\xi)$ is the characteristic mode of a pinned–pinned beam and

$$\psi\left(x\right)=\sin(n\pi x) \quad n=1,2,3,... \tag{21}$$

The substitution of Eq. (20) into Eq. (17) leads to

$$\ddot{q}+\frac{\left(EI+GA\lambda^2\right)\left(n\pi\right)^4}{mL^4\omega^2}q+\frac{EA\left(n\pi\right)^4}{4mL^2\omega^2}q^3=0 \tag{22}$$

where $\dot{(\ )}=\partial(\ )/\partial\hat{t}$ and $\hat{t}=\omega t$.

$$\omega=\left[\frac{\left(EI+GA\lambda^2\right)\left(n\pi\right)^4}{mL^4}\right]^{1/2} \tag{23}$$

one obtains

$$\ddot{q}+q+\gamma q^3=0 \tag{24}$$

in which the dimensionless parameter γ is given by

$$\gamma=\frac{AEL^2}{4\left(EI+GA\lambda^2\right)}=\frac{k}{2}$$
$$=\frac{3L^2}{h^2\left\{1+\left[6/(1+\mu)\right]\times\left(\lambda/h\right)^2\right\}}=\frac{3L^2}{h^2}\beta \tag{25}$$

It is worth noting that Eq. (24) is a classical Duffing-type equation which represents a nonlinear oscillator without damping. This equation may be solved via various methods, such as the method of harmonic balance, equivalent linearization, generalized averaging and multiple scales method [38]. By multiplying (24) by $\dot{q}$ and integrating with respect to time, the following energy balance equation is obtained

$$\dot{q}^2+q^2+\frac{1}{2}\gamma q^4=H=\text{constant}. \tag{26}$$

The constant H is evaluated from initial conditions.
By assuming initial conditions (19), one has

$$H=w_{\max}^2+\frac{1}{2}\gamma w_{\max}^4 \tag{27}$$

Putting H into (26) leads to

$$\dot{q}^2=\left(w_{\max}^2-q^2\right)+\frac{1}{2}\gamma\left(w_{\max}^4-q^4\right) \tag{28}$$

By introducing new parameters χ_1 and χ_2 in the following way

$$\chi_1^2=1+\gamma, \quad \chi_2^2=\frac{\gamma}{2\chi_1^2} \tag{29a,b}$$

Such that the differential equation has solutions in terms of Jacobi elliptic function. Hence, Eq. (28) can be rewritten as follows:

$$\begin{aligned}\dot{q}^2&=w_{\max}^2-q^2+\frac{\gamma}{2}w_{\max}^4-\frac{\gamma}{2}q^4\\&=(\chi_1^2-2\chi_2^2\chi_1^2)(w_{\max}^2-q^2)+\chi_1^2\chi_2^2(w_{\max}^4-q^4)\\&=(w_{\max}^2-q^2)(\chi_1^2-2\chi_2^2\chi_1^2+\chi_1^2\chi_2^2(w_{\max}^2+q^2))\\&=(w_{\max}^2-q^2)(1+\chi_1^2\chi_2^2(w_{\max}^2+q^2))\end{aligned} \tag{30}$$

and it reduces to

$$\left(\frac{dq}{dK}\right)^2=\left(w_{\max}^2-q^2\right)\left(\chi_2^2q^2-\chi_2^2+w_{\max}^2\right). \tag{31}$$

Then, assuming $q=\cos\varphi$ we can obtain Jacobi elliptic function [40] with the modulus k, defined by Eq. (29b),

$$K=\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-\chi_2^2\sin^2\varphi}} \tag{32}$$

From the inversion of Eq. (31), the solution for q can be obtaion as follows:

$$q=cn[K,k] \tag{33}$$

The period of the function cn[K,k] is 4K and is defined by using the complete elliptic integral,

$$4K=4\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-\chi_2^2\sin^2\varphi}} \tag{34}$$

Then, the corresponding frequency for this nonlinear problem for each mode is defined by using the following equation:

$$\omega_{nl}=\frac{\pi\sqrt{1+w_{\max}^2\gamma}}{2K} \tag{35}$$

4. Numerical Results

Figure 1 plot the nonlinear fundamental frequency ratio versus dimensionless amplitude curves for beam. Beam exhibit typical hardening behavior, i.e., the nonlinear frequency ratio increases as the vibration amplitude is increased.

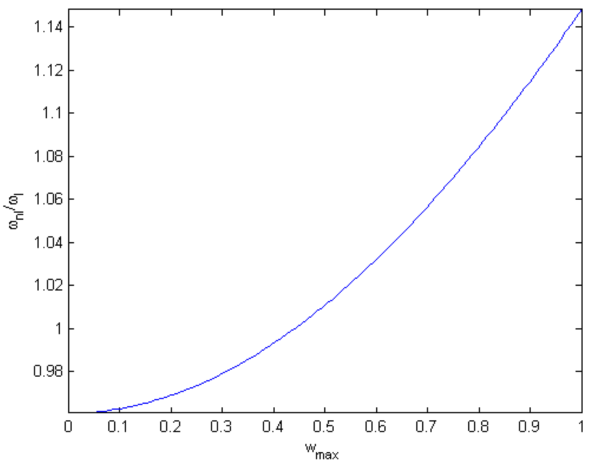

Fig. 1 Nonlinear frequency ratio versus dimensionless amplitude curves for beams

Fig. 2 shows how frequency ratios, μ, predicted by the non-classical beam theory change with the beam thickness (or h/ℓ).

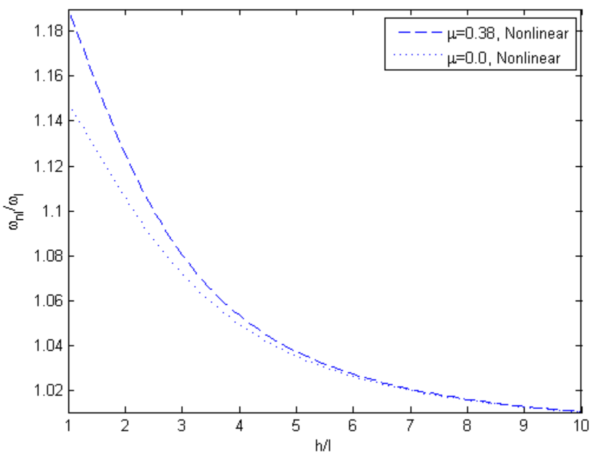

Fig. 2. The ratio of the non-classical frequency to the linear classical frequency as a function of h/ℓ.

Figure 3 displays the phase plane diagrams (q versus $\dot{q}$) for beam.

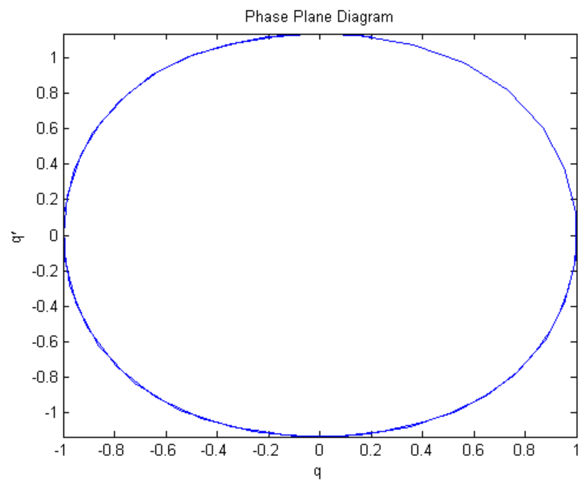

Fig. 3. Phase plane diagram for nonclassical microscale beam.

Figure 4 gives dimensionless vibration amplitude as a function of dimensionless time for beams

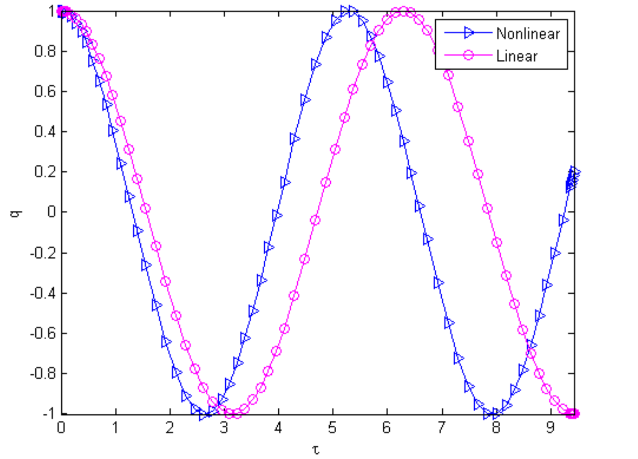

Fig. 4. Time history of dimensionless amplitudes for non-classical microscale beams

The results displayed in Fig. 2 are obtained from Eq. (35), for non-classical beam model. To illustrate the Poisson effect, two different values of Poisson's ratio, i.e., $\mu = 0$ and $\mu = 0.38$ are used. It is worth noting that, when h/ℓ is approximately equal to one, the size effect is remarkably visible.

5. Conclusions

A nonlinear non-classical Euler–Bernoulli beam model with an energy formulation to study the vibration behavior of microscale beams on immovable ends has been developed. In the present nonlinear model, the nonlinearity associated with the internal material length scale constant considered. The analysis is performed within the context of non-classical continuum mechanics. For the free vibration of a microscale beam, it is found that the nonlinear frequencies are much higher than the linear ones. This finding is very useful for high-performance microresonators in which one of the basic requirements is to match the frequencies of signals of interest. Therefore, compared with the linear non-classical beam model, the nonlinear non-classical model and its conclusions regarding vibration properties may be more reliable. The results obtained in this paper highlight the importance of considering nonlinearity and size effects in the proper design of microscale devices and systems such as biosensors, atomic force microscopes and MEMS.

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