Invex Energies on Riemannian Manifolds

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Abstract: In this paper we study the Riemannian convexity of energy functionals connected to the volumetric energy and the kinetic energy. Some of the Lagrangians we discuss about have an interesting origin: they are obtained when considering Euler-Lagrange prolongations of a PDE system of order one, such that, the solutions of this system are ultra-potential maps. An important result consists in proving that these geometrical functionals are always invex, while their convexity is restricted by the geometry of the underlying manifolds.
Section 1 contains some historical and bibliographical notes. Section 2 studies the volumetric energy, the kinetic energy and least squares type energies. Section 3 uses the geodesic deformations to study the Riemannian convexity of functionals. Section 4 analyses some convex functions generated by convex functionals. Section 5 proves the Riemannian invexity of some geometrical energies. Section 6 contains open problems regarding thin-plate spline energy, pairs of type (Lagrangian, Hamiltonian) and energy-momentum tensor field.

Keywords: Energy functional; volumetric energy; pairing map; invexity; Riemannian convexity.
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1. INTRODUCTION

Surely the theory of minimal submanifolds [2]-[4] and the theory of harmonic maps [1] are amongst the simplest and yet general intrinsic variational problems of Riemannian geometry. Being involved in the subject, our research group in University Politehnica of Bucharest changed recently the direction of research, from static variational calculus to optimal dynamics, formulating and studying multitime optimal controlled problems whose solutions are minimal submanifolds or harmonic (potential) maps (see [13]-[24]). This direction gives new interpretations and geometric descriptions of the solutions and creates generalizations. Another dynamic approach of minimal submanifolds appears in the papers [6], [8].

This paper provides a general principle how to define energies for manifold-valued mappings between Riemannian manifolds and for least squares approximations of maps. It has three roots: minimal submanifolds theory, harmonic or potential maps theory and geometric dynamics [11], [12], [17], [21], [22].

The first idea, in Section 2, is to define and study the smallest volumetric energy submanifolds. Passing from the volume functional to the volumetric energy has practical reasons, since it refers to a smooth Lagrangian which is important in the areas of molecular engineering and materials sciences due to anticipated nanotechnology applications. The second idea, refers to the kinetic energies and some energy functionals used to extend PDE systems of order one to Euler-Lagrange PDE systems of order two.

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The idea of considering Euler-Lagrange prolongations of PDE systems of first order was announced in [9], [11] and analyzed afterwords in ([7], [11]-[24]) . We reconsider again this problem, giving original proofs. Moreover, we also consider a new extension and we prove that, there are certain geometric configurations such that the solutions of the initial PDE system to minimize the total energy.

Section 3 gives important results regarding the Riemannian convexity of functionals based on geodesic deformations. A method to obtain Riemannian convex functions using Riemannian convex energies is derived in Section 4 (see also [5], [10]). Section 5 defines and studies the invexity of the volumetric and kinetic energy functionals relative to the Riemannian structures of the involved manifolds. The invexity proves a strong correlation between the convex nature of the Lagrangian and the convexity of the action itself.

2. ENERGIES ON RIEMANNIAN MANIFOLDS

In this section, we use a compact m-dimensional Riemannian manifold (N, h) and an n-dimensional Riemannian manifold (M, g).

Lemma 1. Let M_1 and M_2 be two differentiable manifolds. If $T \in \mathcal{T}_p^0(M_2)$ is a tensor field on M_2, $x : M_1 \to M_2$ is a differentiable map and $\varphi : M_1 \times (-\delta, \delta) \to M_2$ is a deformation of $x(\cdot)$, then

$$\frac{d}{d\epsilon}\varphi_\epsilon^* T|_{\epsilon=0} = x^*(Y(T)), \tag{1}$$

where $\varphi_\epsilon : M_1 \to M_2$, $\varphi_\epsilon(t) = \varphi(t, \epsilon)$, $Y = \varphi_* \frac{\partial}{\partial \epsilon}$ and $Y(T)$ denotes the Lie derivative of the tensor field T with respect to the vector field Y.

In particular, the previous result is valid for differential forms, too.

Lemma 2. With the same hypotheses as above, the vector fields $Y = \varphi_* \frac{\partial}{\partial \epsilon}$ and $Z_\gamma = \varphi_* \frac{\partial}{\partial t^\gamma}$ satisfy the relations

$$[Y, Z_\gamma] \circ x(\cdot) = 0, \ \forall \gamma \in \overline{1, m}. \quad (2)$$

Proof. We can interpret the domain of φ as a product manifold. Therefore

$$[Y, Z_\gamma] \circ x(\cdot) = [\varphi_* \frac{\partial}{\partial \epsilon}, \varphi_* \frac{\partial}{\partial t^\gamma}] \circ x(\cdot) = \varphi_* [\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial t^\gamma}]|_{\epsilon=0} = 0.$$

□

Remark The immediate consequence of the previous two results is that

$$\frac{d}{d\epsilon}[\varphi_\epsilon^* T]_{\alpha_1 \dots \alpha_p}(t)|_{\epsilon=0} = Y(T(Z_{\alpha_1}, ..., Z_{\alpha_p}))(x(t)), \ \forall t \in M_1.$$

The most important geometrical functionals, that we use in this paper, are contained in the next two definitions.

Definition Let $x : N \to M$ be a fixed m-sheet.

1) The multiple integral

$$J[x(\cdot)] = \frac{1}{2} \int_N (det(x^* g))(t) dt \quad (3)$$

is called *the volumetric energy of the submanifold map* $x(\cdot)$.

2) The multiple integral

$$\overline{J}[x(\cdot)] = \frac{1}{2} \int_N Tr^h (x^* g)(t) \sqrt{h(t)} dt \quad (4)$$

is called *the kinetic energy of the submanifold map* $x(\cdot)$.

New important energies are used to build Euler-Lagrange prolongations of a PDE system of order one or least squares approximations of maps (see [7], [11], [12], [17], [21]). To describe them, we start with a distinguished tensor field $X = X^i_\alpha(t, x) \frac{\partial}{\partial x^i} \otimes dt^\alpha$ on $N \times M$, satisfying the complete integrability conditions

$$\frac{\partial X^i_\alpha}{\partial t^\beta} + \frac{\partial X^i_\alpha}{\partial x^j} X^j_\beta = \frac{\partial X^i_\beta}{\partial t^\alpha} + \frac{\partial X^i_\beta}{\partial x^j} X^j_\alpha,$$

and we consider the PDE system of order one

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t)), \quad (5)$$

where $x(\cdot)$ denotes an m-sheet, that is, $x : N \to M$.

From now on, we treat N and M as the horizontal, respectively the vertical components of $N \times M$ and we identify the lifts of vector fields with the original vector fields, using the notations U, V,..., for the horizontal ones, and Y, Z,..., for the vertical ones. We also denote by ∇ the Levi-Civita connection on $N \times M$. We introduce a (1,2)-tensor field

$$F : \mathcal{X}(N) \times \mathcal{X}(M) \to \mathcal{X}(M),$$

$$g(F(U)(Y), Z) = g(\nabla_Y (X(U)), Z) - g(\nabla_Z (X(U)), Y), \quad (6)$$

$\forall U \in \mathcal{X}(N)$, $\forall Y, Z \in \mathcal{X}(M)$. If $x(\cdot)$ is a solution of the PDE system (5), we introduce the map

$$\overline{x} : N \to N \times M, \ \overline{x}(t) = (t, x(t)).$$

Our purpose is to extend the system (5) to a PDE system of order two representing the Euler-Lagrange equations associated to a suitable action. The first step is to consider the PDEs (5) as a relation between tensor fields along $\overline{x}(\cdot)$ on $N \times M$, that is

$$x_* \circ \overline{x}(\cdot) = X \circ \overline{x}(\cdot),$$

where x_* is the differential of the map $x(\cdot)$. If $U \in \mathcal{X}(N)$, then $\overline{x}_*(U) = (U, x_*(U))$ and when we differentiate the previous relation with respect to U, we obtain

$$\begin{aligned} 0 = & g(\nabla_U[(x_* - X)(V)], Y) - g((x_* - X)(\nabla_U V), Y) \\ & + g(\nabla_{x_* U} x_* V, Y) - g(F(V)(x_* U), Y) \\ & - g(\nabla_Y (X(V)), x_* U). \end{aligned} \quad (7)$$

Successively, we modify the last two terms in the PDE system (7), obtaining three extensions:

$$\begin{aligned} 0 = & g(\nabla_U[(x_* - X)(V)], Y) - g((x_* - X)(\nabla_U V), Y) \\ & + g(\nabla_{x_* U} x_* V, Y) - g(F(V)(x_* U), Y) \\ & - g(\nabla_Y (X(V)), X(U)); \end{aligned} \quad (8)$$

$$\begin{aligned} 0 = & g(\nabla_U[(x_* - X)(V)], Y) - g((x_* - X)(\nabla_U V), Y) \\ & + g(\nabla_{x_* U} x_* V, Y) - g(F(V)(X(U)), Y) \\ & - g(\nabla_Y (X(V)), X(U)); \end{aligned} \quad (9)$$

$$\begin{aligned} 0 = & g(\nabla_U[(x_* - X)(V)], Y) - g((x_* - X)(\nabla_U V), Y) \\ & + g(\nabla_{x_* U} x_* V, Y) - g(F(V)(X(U)), Y) \\ & - g(\nabla_Y (X(V)), x_*(U)). \end{aligned} \quad (10)$$

Let E denote the set of m-sheets from N to M and $J : E \to I\!R$, $x(\cdot) \to J[x(\cdot)]$ be an associated functional. If $x(\cdot) \in E$ and $\varphi : N \times (-\delta, \delta) \to M$ is a deformation of $x(\cdot)$, let $Y \in \mathcal{X}_{x(\cdot)}(M)$ be the infinitesimal deformation of φ along $x(\cdot)$.

Definition The operator

$$dJ_{x(\cdot)} : \mathcal{X}_{x(\cdot)}(M) \to I\!R, \ dJ_{x(\cdot)}[Y] = \frac{d}{d\epsilon} J[\varphi(\cdot, \epsilon)]|_{\epsilon=0}$$

is called *the differential of the functional* J *at* $x(\cdot)$.

Theorem 3. The solutions of the PDE system (8) are critical points for the functional

$$\begin{aligned} J[x(\cdot)] = \frac{1}{2} \int_N & h^{\alpha\beta}(t) g_{ij}(x(t)) [x^i_\alpha(t) - X^i_\alpha(t, x(t))] \\ & [x^j_\beta(t) - X^j_\beta(t, x(t))] \sqrt{h(t)} dt. \end{aligned} \quad (11)$$

Proof. If $\varphi : N \times (-\delta, \delta) \to M$ is a deformation of $x(\cdot)$, then $\overline{\varphi} : N \times (-\delta, \delta) \to N \times M$, $\overline{\varphi}(t, \epsilon) = (t, \varphi(t, \epsilon))$ is a deformation of $\overline{x}$ and it follows that

$$Y = \overline{\varphi}_* \frac{\partial}{\partial \epsilon} = (0, \varphi_* \frac{\partial}{\partial \epsilon}) = (0, Y(\varphi(t, \epsilon))),$$

that is , Y can be identified with the vector field tangent to the deformation φ on M. We also consider the vector fields

$$Z_\alpha = \varphi_* \frac{\partial}{\partial t^\alpha}$$

and

$$\overline{Z_\alpha} = \overline{\varphi}_* \frac{\partial}{\partial t^\alpha} = (\partial_\alpha, Z_\alpha).$$

If we take $U = \partial_\alpha$, $V = \partial_\beta$ and $Y = \partial_i$ in (8), then we obtain

$$0 = g(\nabla_{\partial_\alpha}(Z_\beta - X_\beta), \partial_i) - H^\gamma_{\alpha\beta} g(Z_\gamma - X_\gamma, \partial_i)$$
$$+ g(\nabla_{Z_\alpha}(Z_\beta - X_\beta), \partial_i) + g(\nabla_{\partial_i} X_\beta, Z_\alpha - X_\alpha).$$

Now, since the critical points of the functional J satisfy the condition $dJ_{x(\cdot)}[Y] = 0,\ \forall Y \in \mathcal{X}_{x(\cdot)}(M)$, we need to compute this differential. Using Lemma (1) and Lemma (2) and the Remark derived from them, we obtain

$$dJ_{x(\cdot)}[Y] = \frac{1}{2}\frac{d}{d\epsilon}\int_N h^{\alpha\beta}\sqrt{h}[(x^*g)_{\alpha\beta} - 2\overline{x}^*(g(\cdot, X_\beta))_\alpha$$
$$+ g(X_\alpha, X_\beta) \circ \overline{x}]dt|_{\epsilon=0}$$
$$= \frac{1}{2}\int_N h^{\alpha\beta}\sqrt{h}[Y(g(Z_\alpha, Z_\beta)) - 2Y(g(Z_\alpha, X_\beta))$$
$$+ Y(g(X_\alpha, X_\beta))]dt$$
$$= \frac{1}{2}\int_N h^{\alpha\beta}\sqrt{h}[Y(g(Z_\alpha - X_\alpha, Z_\beta - X_\beta))]dt$$
$$= \int_N h^{\alpha\beta}\sqrt{h}[g(\nabla_Y(Z_\alpha - X_\alpha), Z_\beta - X_\beta)]dt$$
$$= \int_N h^{\alpha\beta}\sqrt{h}[-g(\nabla_Y X_\alpha, Z_\beta - X_\beta)$$
$$+ g(\nabla_{Z_\alpha} Y, Z_\beta - X_\beta)]dt$$
$$= \int_N h^{\alpha\beta}\sqrt{h}[-g(\nabla_Y X_\alpha, Z_\beta - X_\beta)$$
$$+ D_\alpha(g(Y, Z_\beta - X_\beta)) - g(Y, \nabla_{Z_\alpha}(Z_\beta - X_\beta))$$
$$- g(Y, \nabla\partial\alpha(Z_\beta - X_\beta))]dt$$
$$= \int_N \{h^{\alpha\beta}\sqrt{h}[-g(\nabla_Y X_\alpha, Z_\beta - X_\beta)$$
$$- g(Y, \nabla_{Z_\alpha}(Z_\beta - X_\beta)) - g(Y, \nabla\partial\alpha(Z_\beta - X_\beta)]$$
$$+ D_\alpha[h^{\alpha\beta}\sqrt{h} g(Y, Z_\beta - X_\beta)]$$
$$- D_\alpha(h^{\alpha\beta}\sqrt{h}) g(Y, Z_\beta - X_\beta)\}dt$$
$$= \int_N h^{\alpha\beta}\sqrt{h}[-g(\nabla_Y X_\alpha, Z_\beta - X_\beta)$$
$$- g(Y, \nabla_{Z_\alpha}(Z_\beta - X_\beta)) - g(Y, \nabla\partial\alpha(Z_\beta - X_\beta))$$
$$+ H^\gamma_{\alpha\beta} g(Y, Z_\gamma - X_\gamma)]dt = 0,\ \forall Y \in \mathcal{X}_{x(\cdot)}(M).$$

Therefore, the Euler-Lagrange PDEs write

$$0 = h^{\alpha\beta}[g(\nabla_{\partial_i} X_\alpha, Z_\beta - X_\beta) + g(\partial_i, \nabla_{\partial_\alpha}(Z_\beta - X_\beta))$$
$$+ g(\partial_i, \nabla_{Z_\alpha}(Z_\beta - X_\beta)) - H^\gamma_{\alpha\beta} g(\partial_i, Z_\gamma - X_\gamma)].$$

□

Remark 1) If the complete integrability conditions of the PDE system (5) are not satisfied (i.e., the system has no solution), then the Euler-Lagrange PDEs (8) are not prolongations.

2) The expression of the functional in Theorem (3) suggests us the idea of considering a similar multiple integral defined using the determinant, that is, a volumetric energy

$$J[x(\cdot)] = \frac{1}{2}\int_N det(\tilde{g}_{\alpha\beta}(t))dt, \tag{12}$$

where

$\tilde{g}_{\alpha\beta}(t) = g_{ij}(x(t))[x^i_\alpha(t) - X^i_\alpha(t, x(t))][x^j_\beta(t) - X^j_\beta(t, x(t))]$.

Theorem 4. If N is an one-dimensional or a two-dimensional compact manifold (i.e., $m = 1$ or $m = 2$), then there are two Riemannian structures, h on N and g on M, and a family of potential maps $V_{\alpha\beta} = V_{\alpha\beta}(t, x)$ on $N \times M$ such that

(1) $\mathrm{grad}^g(V_{\alpha\beta}) = [\nabla_{\partial_\alpha + X_\alpha} X](\partial_\beta),\ \forall \alpha, \beta = \overline{1, m}$ and
(2) the solutions of the extended system (9) are critical points for the functional

$$J[x(\cdot)] = \int_N h^{\alpha\beta}(t)[\frac{1}{2} g_{ij}(x(t)) x^i_\alpha(t) x^j_\beta(t)$$
$$+ V_{\alpha\beta}(t, x(t))]\sqrt{h(t)}dt.$$

Proof. For the first statement of the theorem, we remark that we need to have solutions for a PDE system of first order with $\frac{n(n+1)}{2} + \frac{m(m+1)}{2} + m^2$ variables and $m^2 n$ conditions. If $m = 1$ or $m = 2$, then the number of the variables exceeds the number of constraints, therefore we have solutions. Otherwise, we have to give inferior limits for n; for example, if $m = 3$, then the dimension n of the manifold M must be at least 15.

Rewriting the second extension for $U = \partial_\alpha$, $V = \partial_\beta$, $Y = \partial_i$, we obtain

$$0 = g(\nabla_{\partial_\alpha}(Z_\beta - X_\beta), \partial_i) - H^\gamma_{\alpha\beta} g(Z_\gamma - X_\gamma, \partial_i)$$
$$+ g(\nabla_{Z_\alpha} Z_\beta, \partial_i) - g(\nabla_{X_\alpha} X_\beta, \partial_i).$$

On the other side,

$$dJ_{x(\cdot)}[Y] = \int_N h^{\alpha\beta}\sqrt{h}[g(\nabla_Y Z_\alpha, Z_\beta) + g(\mathrm{grad} V_{\alpha\beta}, Y)]dt$$
$$= \int_N D_\alpha[h^{\alpha\beta}\sqrt{h} g(Y, Z_\beta)] + h^{\alpha\beta}\sqrt{h}[H^\gamma_{\alpha\beta} g(Y, Z_\gamma)$$
$$- g(Y, \nabla_{\partial_\alpha} Z_\beta) - g(Y, \nabla_{Z_\alpha} Z_\beta) + g(\nabla_{\partial_\alpha} X_\beta, Y)$$
$$- H^\gamma_{\alpha\beta} g(Y, X_\gamma) + g(\nabla_{X_\alpha} X_\beta, Y)]dt$$
$$= \int_N h^{\alpha\beta}\sqrt{h}[H^\gamma_{\alpha\beta} g(Y, Z_\gamma) - g(Y, \nabla_{\partial_\alpha} Z_\beta)$$
$$- g(Y, \nabla_{Z_\alpha} Z_\beta) + g(\nabla_{\partial_\alpha} X_\beta, Y) - H^\gamma_{\alpha\beta} g(Y, X_\gamma)$$
$$+ g(\nabla_{X_\alpha} X_\beta, Y)]dt,\ \forall Y \in \mathcal{X}(M).$$

Consequently, the Euler-Lagrange PDEs are

$$0 = h^{\alpha\beta}[g(\partial_i, \nabla_{\partial_\alpha}(Z_\beta - X_\beta)) - H^\gamma_{\alpha\beta} g(\partial_i, (Z_\gamma - X_\gamma))$$
$$+ g(\partial_i, \nabla_{Z_\alpha} Z_\beta) - g(\partial_i, \nabla_{X_\alpha} X_\beta)].$$

□

Open problem Find a Lagrangian such that the Euler-Lagrange equations to be obtained by considering the trace of the expression (10) written in local coordinates:

$$0 = h^{\alpha\beta}[g(\nabla_{\partial_\alpha}(Z_\beta - X_\beta), \partial_i) - H^\gamma_{\alpha\beta} g(Z_\gamma - X_\gamma, \partial_i)$$
$$+ g(\nabla_{Z_\alpha} Z_\beta, \partial_i) - g(\nabla_{X_\alpha} X_\beta, \partial_i) - g(\nabla_{\partial_i} X_\beta, Z_\alpha - X_\alpha)].$$

3. RIEMANNIAN CONVEXITY OF ENERGY FUNCTIONALS

The theory in this Section extends some ideas, from functions to functionals, in the Riemannian language of the paper [10]. For that we need the geodesic deformation map.

Definition A deformation map $\varphi : N \times [0,1] \to M$ is called *geodesic deformation* if $\varphi(t,\cdot)$ is a geodesic in (M,g), for each $t \in N$.

Definition A subset $F \subset E$ is called *totally convex* if, for all pairs of m-sheets $x(\cdot), y(\cdot) \in F$ and all geodesic deformations $\varphi : N \times [0,1] \to M$, $\varphi(\cdot,0) = x(\cdot)$, $\varphi(\cdot,1) = y(\cdot)$, we have

$$\varphi(\cdot,\epsilon) \in F, \ \forall \epsilon \in [0,1].$$

Definition Let $F \subset E$ be a totally convex subset of m-sheets. A functional $J : F \to I\!R$ is called *Riemannian convex* if

$$J[\varphi(\cdot,\epsilon)] \leq (1-\epsilon)J[x(\cdot)] + \epsilon J[y(\cdot)], \tag{13}$$

for all $x(\cdot)$ and $y(\cdot)$ in F, for all geodesic deformations $\varphi : N \times [0,1] \to M$ connecting $x(\cdot)$ and $y(\cdot)$ and for all $\epsilon \in [0,1]$.

Definition The functional J is called *Riemannian strictly convex* if

$$J[\varphi(\cdot,\epsilon)] < (1-\epsilon)J[x(\cdot)] + \epsilon J[y(\cdot)], \tag{14}$$

for all $x(\cdot)$, $y(\cdot)$, φ as above, $x(\cdot) \neq y(\cdot)$ and $\epsilon \in (0,1)$.

Lemma 5. The functional $J : F \to I\!R$ is convex (strictly convex) iff, for each geodesic deformation $\varphi : N \times [0,1] \to M$ lying in F, the function

$$J_\varphi : [0,1] \to I\!R, \ J_\varphi(\epsilon) = J[\varphi(\cdot,\epsilon)]$$

is convex (strictly convex) on the interval $[0,1]$.

3.1 Riemannian convexity of volumetric energy functional

Let $J : E \to I\!R$ be the volumetric energy functional, $x(\cdot) \in E$, $\varphi : N \times [0,1] \to M$ be a geodesic deformation of the map $x(\cdot)$ and $Y = \frac{\partial \varphi}{\partial \epsilon}$ be the vector field tangent to this geodesic deformation. Moreover, we consider the family of vector fields $Z_\alpha = \varphi_* \frac{\partial}{\partial t^\alpha}$ and an orthonormal frame field $\{E_1, ..., E_m\}$ on $\mathcal{X}(x(N))$.

Lemma 6. If the bilinear form

$$\Omega_{x(\cdot)} = (x^*g)^{\alpha\beta}\mathrm{Hess}(g(Z_\alpha, Z_\beta))$$

is positive semidefinite on $\mathcal{X}_{x(\cdot)}(M)$ and if

$$[\sum_{\alpha=1}^{m} Y(g)(E_\alpha, E_\alpha)]^2 - \sum_{\alpha,\beta=1}^{m} [Y(g)(E_\alpha, E_\beta)]^2 \geq 0, \tag{15}$$

$\forall Y \in \mathcal{X}_{x(\cdot)}(M)$, then the volumetric energy functional J is Riemannian convex at $x(\cdot)$. If $\Omega_{x(\cdot)}$ is positive definite, then J is strictly convex at $x(\cdot)$.

Proof. We consider $J_\varphi : [0,1] \to I\!R$, $J_\varphi(\epsilon) = J[\varphi(\cdot,\epsilon)]$. Then J is Riemannian convex at $x(\cdot)$ iff $\frac{d^2 J_\varphi}{d\epsilon^2}|_{\epsilon=0} \geq 0$. Furthermore, if $\frac{d^2 J_\varphi}{d\epsilon^2}|_{\epsilon=0} > 0$, then J is strictly convex at $x(\cdot)$. By direct computation we have

$$\begin{aligned}\frac{d^2\varphi}{d\epsilon^2}(0) = \frac{1}{2}\int_N & det(x^*g)\{[(x^*g)^{\alpha\beta}(x^*g)^{\mu\nu} \\ & - (x^*g)^{\alpha\mu}(x^*g)^{\beta\nu}] \cdot Y(g(Z_\alpha, Z_\beta))Y(g(Z_\mu, Z_\nu)) \\ & + (x^*g)^{\alpha\beta}[Y(Y(g(Z_\alpha, Z_\beta)))]\}dt\end{aligned}$$

The hypotheses ensure us that the first term is positive semidefinite. Therefore

$$\begin{aligned}\frac{d^2\varphi}{d\epsilon^2}(0) &\geq \frac{1}{2}\int_N det(x^*g)(x^*g)^{\alpha\beta}\mathrm{Hess}(g(Z_\alpha, Z_\beta))(Y,Y)dt \\ &= \frac{1}{2}\int_N det(x^*g)\Omega_{x(\cdot)}(Y,Y)(x(t))dt\end{aligned}$$

□

Remark If N is an interval and $x : N \to M$ is a differentiable curve, the relation (15) is naturally satisfied.

Theorem 7. If the induced tensor field

$$R_Y \in \mathcal{T}^0_2(M), \ R_Y(Z,W) = R(Y,Z,Y,W)$$

is negative semidefinite and $Y(g)$ satisfies the relation (15), $\forall Y \in \mathcal{X}(M)$, then the volumetric energy functional is Riemannian convex at $x(\cdot)$.

Proof. For $x(\cdot) \in E$, we consider the family of vector fields $Z_\alpha = \varphi_* \frac{\partial}{\partial t^\alpha}$, with φ a geodesic deformation of $x(\cdot)$. If $Y \in \mathcal{X}(M)$ is the vector field tangent to this geodesic deformation, then

$$\begin{aligned}&\mathrm{Hess}(g(Z_\alpha, Z_\beta))(Y,Y) = Y(Y(g(Z_\alpha, Z_\beta))) \\ &= g(\nabla_Y\nabla_Y Z_\alpha, Z_\beta) + 2g(\nabla_Y Z_\alpha, \nabla_Y Z_\beta) + g(Z_\alpha, \nabla_Y\nabla_Y Z_\beta).\end{aligned}$$

On the other hand, since $[Y, Z_\alpha] = 0$, $\forall \alpha \in \overline{1,m}$ and $\nabla_Y Y = 0$, it follows

$$g(\nabla_Y\nabla_Y Z_\alpha, Z_\beta) = -R(Z_\beta, Y, Z_\alpha, Y),$$

and hence

$$\begin{aligned}\mathrm{Hess}(g(Z_\alpha, Z_\beta))(Y,Y) = &-2R(Z_\beta, Y, Z_\alpha, Y) \\ &+ 2g(\nabla_Y Z_\alpha, \nabla_Y Z_\beta).\end{aligned}$$

We have

$$(x^*g)^{\alpha\beta} R(Z_\beta, Y, Z_\alpha, Y) \circ x = Tr(x^*R_Y) \leq 0$$

and

$$\begin{aligned}(x^*g)^{\alpha\beta} g(\nabla_Y Z_\alpha, \nabla_Y Z_\beta) \circ x &= (x^*g)^{\alpha\beta} g(\nabla_{Z_\alpha} Y, \nabla_{Z_\beta} Y) \circ x \\ &= Tr(x^*\Omega_Y),\end{aligned}$$

where $\Omega_Y = g(\nabla Y, \nabla Y)$ and we get

$$\Omega_{x(\cdot)}(Y,Y) \circ x = -2Tr(x^*R_Y) + 2Tr(x^*\Omega_Y).$$

Proving the fact that Ω_Y is positive semidefinite, $\forall Y \in \mathcal{X}(M)$ is the last step. Indeed,

$$\Omega_Y(Z,Z) = g(\nabla_Z Y, \nabla_Z Y) = \|\nabla_Z Y\|^2 \geq 0$$

and, since $R_Y \leq 0$ it follows that $\Omega_{x(\cdot)}$ is positive semidefinite along $x(\cdot)$ and J is a convex functional. □

3.2 Riemannian convexity of kinetic energy functional

Lemma 8. If the bilinear form

$$\overline{\Omega}_{x(\cdot)} = h^{\alpha\beta}\mathrm{Hess}(g(Z_\alpha, Z_\beta))$$

is positive semidefinite on $\mathcal{X}_{x(\cdot)}(M)$, then the kinetic energy functional $\overline{J}$ is Riemannian convex at $x(\cdot)$. If $\overline{\Omega}_{x(\cdot)}$ is positive definite, then $\overline{J}$ is strictly convex at $x(\cdot)$.

Proof. Similar arguments as in Lemma (6). □

Theorem 9. If the tensor field R_Y is negative semidefinite, $\forall Y \in \mathcal{X}(M)$, then the kinetic energy functional $\overline{J} : E \to I\!R$, $\overline{J}[x(\cdot)] = \frac{1}{2}\int_N Tr^h(x^*g)(t)\sqrt{h(t)}dt$ is convex.

Proof. Again, we obtain

$$\text{Hess}(g(Z_\alpha, Z_\beta))(Y,Y) = -2R(Z_\beta, Y, Z_\alpha, Y) + 2g(\nabla_Y Z_\alpha, \nabla_Y Z_\beta),$$

and

$$\overline{\Omega}_{x(\cdot)}(Y,Y)(x(t)) = -2Tr^h(x^* R_Y) + 2Tr^h(x^* \Omega_Y),$$

with $-R_Y$ and Ω_Y denoting two positive semidefinite bilinear forms. Therefore, $\overline{J}$ is a convex functional. □

4. CONVEX FUNCTIONS GENERATED BY CONVEX FUNCTIONALS

The following theorem gives us a method for creating convex functions on a complete Riemannian manifold (M,g) using convex functionals (see also [10]). Let us consider $N \subset R^m$, t_0 and t_1 two fixed points in N, $E = \{\Phi : N \to M |\ \Phi \text{ is an } m\text{-sheet}\}$ and $J : E \to I\!R$ a functional. If $x_0 \in M$ is a fixed point, let

$$\Gamma_x = \{\Phi \in E |\ \Phi(t_0) = x_0, \Phi(t_1) = x\}.$$

Theorem 10. If $J : E \to I\!R$ is a convex functional, then

$$f : M \to I\!R,\ f(x) = \inf_{\Phi \in \Gamma_x} J[\Phi]$$

is a Riemannian convex function on M.

Proof. Let us consider $x, y \in M$ and $\psi : [0,1] \to M$ a geodesic such that $\psi(0) = x$ and $\psi(1) = y$. We also consider the set

$$\Gamma = \{\varphi : N \times [0,1] \to M |\ \varphi(t,\cdot) \text{ geodesic},\ \forall t \in N,\ \varphi(t_0,\cdot) = x_0,\ \varphi(t_1,0) = x,\ \varphi(t_1,1) = y\}.$$

We have

$$\begin{aligned} f(\psi(\epsilon)) &= \inf_{\Phi \in \Gamma_{\psi(\epsilon)}} J[\Phi] = \inf_{\varphi \in \Gamma, \varphi(t_1,\cdot) = \psi(\cdot)} J[\varphi(\cdot,\epsilon)] \\ &\le (1-\epsilon) \inf_{\varphi} J[\varphi(\cdot,0)] + \epsilon \inf_{\varphi} J[\varphi(\cdot,1)] \\ &= (1-\epsilon) \inf_{\Phi \in \Gamma_x} J[\Phi] + \epsilon \inf_{\Phi \in \Gamma_y} J[\Phi] \\ &= (1-\epsilon) f(x) + \epsilon f(y). \end{aligned}$$

Therefore, f is a convex function. □

In the following we give an example, based on the convexity of the volumetric energy functional.

Let (M,g) be a complete Riemannian manifold and $x_0 \in M$ be a fixed point. If R_X is negative semidefinite on M, $\forall X \in \mathcal{X}(M)$, then $f : M \to I\!R$, $f(x) = d^2(x_0, x)$ is a convex function (a direct prove can be found in [10]). We prove this statement using the previous Theorem. We consider $N = [a,b]$ and the "volumetric" energy functional J. Then,

$$f(x) = \inf_{\gamma \in \Gamma_x} \frac{1}{2} \int_a^b g(\dot\gamma(t), \dot\gamma(t)) dt = \inf_{\gamma \in \Gamma_x} J[\gamma].$$

Since J is a convex functional it follows that $f(x) = d^2(x_0, x)$ is a convex function.

5. INVEXITY OF ENERGIES

This Section extends some ideas from functions (see [5]) to functionals. Let $J^1(N,M)$ denote the jet bundle associated to N and M. A differentiable function L on $J^1(N,M)$ is called *Lagrangian* and the functional defined by multiple integral

$$J : E \to I\!R,\ J[x(\cdot)] = \int_N L(t^\alpha, x^i(t), \frac{\partial x^i}{\partial t^\alpha}(t)) dt$$

is the action associated to L. If $x(\cdot) \in E$, we denote by $\Phi : N \to J^1(N,M)$ the submanifold map $\Phi(t) = (t^\alpha, x^i(t), \frac{\partial x^i}{\partial t^\alpha}(t))$ and we substitute $J[x(\cdot)]$ with $J[\Phi]$. We also denote by F the set off all these submanifold maps.

Definition A vectorial map

$$\eta : F \times F \to \mathcal{X}(J^1(N,M)),\ \eta(\Psi,\Phi) \in \mathcal{X}_\Phi(J^1(N,M)) \tag{16}$$

is called *pairing map on* F.

Remark A more refined concept of invexity can be obtained if we use pairing maps η defined by

$$\begin{aligned} \eta(\Psi,\Phi)(t) = (0, \eta^i(t, x^i(t), y^i(t), x^i_\gamma(t), y^i_\gamma(t)), \\ D_\alpha[\eta^i(t, x^j(t), y^j(t), x^j_\sigma(t), y^j_\sigma(t))]), \end{aligned} \tag{17}$$

where D_α denotes the total derivative with respect to t^α.

Example If $\Phi \in F$ and $V_\Phi = \{\Psi \in F |\ \Psi(t) \in V_{\Phi(t)},\ \forall t \in N\}$, where $V_{\Phi(t)}$ is a neighborhood of $\Phi(t)$ such that $exp_{\Phi(t)} : T_{\Phi(t)} J^1(N,M) \to V_{\Phi(t)}$ is a diffeomorphism, then we consider the map

$$\eta^{(\Phi)} : V_\Phi \to \mathcal{X}_\Phi(J^1(N,M)),\ \eta^{(\Phi)}(\Psi)(t) = exp^{-1}_{\Phi(t)}(\Psi(t)). \tag{18}$$

Furthermore, we denote by η_0 a pairing map satisfying

$$\eta_0(\Psi,\Phi) = \eta^{(\Phi)}(\Psi),\ \forall \Psi \in V_\Phi. \tag{19}$$

If $\eta : F \times F \to \mathcal{X}(J^1(N,M))$ is a pairing map and $\Phi, \Psi \in F$, we consider $\gamma_{\Psi\Phi\eta} : N \times (-\delta,\delta) \to J^1(N,M)$, $[0,1] \subset (-\delta,\delta)$, a geodesic deformation satisfying

$$\gamma_{\Psi\Phi\eta}(t,0) = \Phi(t),\ \forall t \in N$$

and

$$\frac{\partial \gamma_{\Psi\Phi\eta}}{\partial \epsilon}(t,0) = \eta(\Psi,\Phi)(t),\ \forall t \in N.$$

Definition Let $\eta : F \times F \to \mathcal{X}(J^1(N,M))$ be a pairing map on F. A functional $J : F \to I\!R$ is called *η-convex* at $\Phi \in F$ if

$$J[\Psi] - J[\Phi] \ge dJ(\Phi)[\eta(\Psi,\Phi)],\ \forall \Psi \in F. \tag{20}$$

The functional J is called *strictly η-convex* at $\Phi \in F$ if

$$J[\Psi] - J[\Phi] > dJ(\Phi)[\eta(\Psi,\Phi)],\ \forall \Psi \in F, \Psi \ne \Phi. \tag{21}$$

Definition The functional J is called *invex* if there is a pairing map $\eta : F \times F \to \mathcal{X}(J^1(N,M))$ such that J is η-convex.

Remark When taking $\eta = \eta_0$, the η-convexity is in fact the Riemannian convexity from the previous sections.

Theorem 11. A functional $J : F \to I\!R$ is invex iff all its critical points are global minimum points.

Theorem 12. If the Lagrangian L is an invex function on $J^1(N,M)$, then the action J is an invex functional.

5.1 The volumetric energy functional

Let N be a compact m-dimensional Riemannian manifold with local coordinates $(t^1, ..., t^m)$, let (M,g) be a complete n-dimensional Riemannian manifold with local coordinates $(x^1, ..., x^n)$ and E be the set of all submanifold maps from N to M.

We have already proved that, in additional hypotheses concerning the Riemannian structure g, the volumetric energy is Riemannian η_0-convex. The next Theorem eliminates this supplementary condition and ensures us about the invexity of this functional.

Theorem 13. The volumetric energy functional is invex.

Proof. We introduce the following differentiable functions on $J^1(N,M)$:

$$\overline{g}_{\alpha\beta}(t^\gamma, x^i, x^i_\gamma) = g_{ij}(x)x^i_\alpha x^j_\beta;\ \overline{g}(t^\gamma, x^i, x^i_\gamma) = det(\overline{g}_{\alpha\beta}).$$

The Lagrangian corresponding to the volumetric energy functional is

$$L(t^\gamma, x^i, x^i_\gamma) = \frac{1}{2}\,\overline{g}(t^\gamma, x^i, x^i_\gamma)$$

and

$$Crit\, L = Crit\,\overline{g} = \{(t,x,0)|\ t\in N,\ x\in M\},$$

and, because all these critical points are also minimum points, it follows that the Lagrangian is invex and, consequently, the functional J is invex.

5.2 The kinetic energy functional

In this subsection, we need a Riemannian metric h on N. Same as before, the kinetic energy is η_0-convex, when the bilinear form R_Y is negative semidefinite.

Theorem 14. The kinetic energy functional is invex.

Proof. The Lagrangian (the energy density) associated to this functional is

$$L(t^\gamma, x^i, x^i_\gamma) = \frac{1}{2}\, h^{\alpha\beta}(t)g_{ij}(x)x^i_\alpha x^j_\beta$$

and

$$Crit\, L = \{(t,x,0)|\ t\in N,\ x\in M\}.$$

Since all the critical points of L are also minimum points it follows that L is an invex Lagrangian and J is an invex functional.

5.3 Invexity of least squares Lagrangian

In the first chapter, we obtained some new functionals (deformations of the volumetric and kinetic energy functionals), when we considered Euler-Lagrange prolongations for a PDE system of first order (see [7], [9]-[24]). Let $X = X^i_\alpha(t,x)\frac{\partial}{\partial x^i}\otimes dt^\alpha$ be a distinguished tensor field on $N\times M$, satisfying the integrability conditions

$$\frac{\partial X^i_\alpha}{\partial t^\beta} + \frac{\partial X^i_\alpha}{\partial x^j}X^j_\beta = \frac{\partial X^i_\beta}{\partial t^\alpha} + \frac{\partial X^i_\beta}{\partial x^j}X^j_\alpha,$$

and we consider the PDE system of order one

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t)),$$

where $x(\cdot)$ denotes an m-sheet, that is, $x : N \to M$. We also introduce the functionals

$$J_X, \overline{J}_X : E \to I\!R,$$

$$J_X[x(\cdot)] = \frac{1}{2}\int_N det(\tilde{g}_{\alpha\beta}(t))dt,$$

where

$$\tilde{g}_{\alpha\beta}(t) = g_{ij}(x(t))[x^i_\alpha(t) - X^i_\alpha(t,x(t))][(x^j_\beta(t) - X^j_\beta(t,x(t))]$$

and

$$\overline{J}_X[x(\cdot)] = \frac{1}{2}\int_N h^{\alpha\beta}(t)\tilde{g}_{\alpha\beta}(t)\sqrt{h(t)}dt.$$

Theorem 15. The functionals J_X and $\overline{J}_X$ are invex.

Proof. We consider the vector field

$$T\in\mathcal{X}(J^1(N,M)),\ T = [x^i_\alpha - X^i_\alpha(t,x)]\frac{\partial}{\partial x^i_\alpha}.$$

If $\overline{L} = \frac{1}{2}\sqrt{h(t)}h^{\alpha\beta}(t)g_{ij}(x)(x^i_\alpha - X^i_\alpha(t,x))(x^j_\beta - X^j_\beta(t,x))$ is the Lagrangian corresponding to $\overline{J}_X$, then L can be also written as

$$L(t^\gamma, x^i, x^i_\gamma) = \frac{1}{2}\,G(T,T)\sqrt{h}\geq 0,$$

where $G = h + g + h^{-1}\otimes g$ is the Riemannian metric induced by h and g on $J^1(N,M)$.

$$Crit\, L = \{(t,x,X^i_\alpha)|\ t\in N,\ x\in M\}$$

and, since all this points are global minimum points it follows that $\overline{J}_X$ is an invex functional.

If we consider now the family of vector fields

$$T_\alpha\in\mathcal{X}(J^1(N,M)),\ T_\alpha = [x^i_\alpha - X^i_\alpha(t,x)]\frac{\partial}{\partial x^i},$$

then the Lagrangian L associated to the functional J_X writes

$$L(t^\gamma, x^i, x^i_\gamma) = \frac{1}{2}\,det(G(T_\alpha, T_\beta))\geq 0.$$

The set of critical points of L is

$$Crit\, L = \{(t,x,X^i_\alpha)|\ t\in N,\ x\in M\}$$

and again, they are minimum points and the functional is invex.

6. OPEN PROBLEMS

6.1 Thin-plate spline energy

Let $x = (x^i) : N\to M,\ t\to x(t)$ and

$$(\mathrm{Hess}\, x)^i_{\alpha\gamma} = \frac{\partial^2 x^i}{\partial t^\alpha\partial t^\gamma} - \Gamma^\lambda_{\alpha\gamma}(t)x^i_\lambda + \Gamma^i_{jk}(x)x^j_\alpha x^k_\gamma.$$

The map x converts geodesics of (N,h) into geodesics of (M,g) if and only if $(\mathrm{Hess}\, x)^i_{\alpha\gamma} = 0$. Analyze the extremals of the second order Lagrangian

$$L = \frac{1}{2}\,h^{\alpha\beta}(t)h^{\gamma\delta}(t)g_{ij}(x)(\mathrm{Hess}\, x)^i_{\alpha\gamma}(\mathrm{Hess}\, x)^j_{\beta\delta},$$

which reduces to the *thin-plate spline energy* in the Euclidean case.

6.2 Pairs of type (Lagrangian, Hamiltonian)

Let $L = L(t,x,x_\alpha))$ be a *Lagrangian* or a *kinetic potential.* If the equations $p^\alpha_i = \dfrac{\partial L}{\partial x^i_\alpha}(t,x,x_\alpha)$ defines an implicit bijection $x_\alpha\to p^\alpha$, then the formula

$$H = x^i_\alpha\frac{\partial L}{\partial x^i_\alpha} - L \tag{21}$$

represents a duality between a Lagrangian and a Hamiltonian, and produces a duality between the Euler-Lagrange PDEs

$$\frac{\partial L}{\partial x^i} - D_\alpha \frac{\partial L}{\partial x^i_\alpha} = 0$$

and the Hamilton PDEs

$$\frac{\partial x^i}{\partial t^\alpha}(t) = \frac{\partial H}{\partial p^\alpha_i}(t, x(t), p(t)),\ \frac{\partial p^\alpha_i}{\partial t^\alpha}(t) = -\frac{\partial H}{\partial x^i}(t, x(t), p(t)).$$

Making abstraction of $\sqrt{\det(h)}$, using the formula (21) and preserving the independent variables t, x, x_α, we can introduce some pairs of functions (L, H) (see also [21]):

1) $$L = \frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - h^{\alpha\beta}g_{ij}x^i_\alpha X^j_\beta,$$
$$H = \frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta;$$

2) $$L = \frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - \frac{1}{2}h^{\alpha\beta}g_{ij}X^i_\alpha X^j_\beta,$$
$$H = \frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta + \frac{1}{2}h^{\alpha\beta}g_{ij}X^i_\alpha X^j_\beta;$$

3) $L = \dfrac{h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta h^{\gamma\delta}g_{kl}X^k_\gamma X^l_\delta}{h^{\lambda\mu}g_{mn}x^m_\lambda X^n_\mu},\ H = -L;$

4) $L = h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta h^{\gamma\delta}g_{kl}X^k_\gamma X^l_\delta - (h^{\lambda\mu}g_{mn}x^m_\lambda X^n_\mu)^2,$
$$H = -(h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta)^2;$$

5) $L = \dfrac{h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta h^{\gamma\delta}g_{kl}X^k_\gamma X^l_\delta}{(h^{\lambda\mu}g_{mn}x^m_\lambda X^n_\mu)^2},\ H = -2L;$

6) $L = \det(g_{ij}x^i_\alpha x^j_\beta),\ H = (m-1)L;$

7) $L = \det\left(\dfrac{1}{2}g_{ij}(x^i_\alpha X^j_\beta + x^i_\beta X^j_\alpha)\right),\ H = (m-1)L.$

All the Lagrangians in this paper are algebraic functions of the following arguments

$$u = h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta,\ v = h^{\alpha\beta}g_{ij}x^i_\alpha X^j_\beta,\ w = h^{\alpha\beta}g_{ij}X^i_\alpha X^j_\beta,$$
$$r = \det(g_{ij}x^i_\alpha x^j_\beta),\ s = \det\left(\frac{1}{2}g_{ij}(x^i_\alpha X^j_\beta + x^i_\beta X^j_\alpha)\right).$$

The pairs (L, H) contain two geometrical ingredients: the *length* and the *angle*. Of course, since $h_{\alpha\beta}, g_{ij}$ are positive definite metrics, we can obtain inequalities satisfied by L or H.

Write the previous Hamiltonians as functions of $p = (p^\alpha_i)$. If we use the same independent variables, how we distinguish between a Lagrangian and a Hamiltonian.

6.3 Least squares Hamiltonian

Our *geometric dynamics* [7], [11], [12], [17], [21] used a least squares Lagrangian L (square of the length) and its associated Hamiltonian (scalar product)

$$H = \frac{1}{2}h^{\alpha\beta}g_{ij}(x^i_\alpha - X^i_\alpha(t,x))(x^j_\beta + X^j_\beta(t,x))\sqrt{\det h}.$$

Of course, this is not conserved along the extremals of L or along the solutions of Hamilton PDEs. On the other hand, we remark that H corresponds either to the pair $\big(x^i_\alpha(t), X^i_\alpha(t, x(t))\big)$ or to the pair $\big(x^i_\alpha(t), -X^i_\alpha(t, x(t))\big)$. If we use simultaneously the PDE system

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha(t, x(t))$$

and the PDE system

$$\frac{\partial x^i}{\partial t^\alpha} = -X^i_\alpha(t, x(t)),$$

then the manifold T must be star-like with respect to the origin. Also, if X^i_α are changed into $-X^i_\alpha$, then the Lagrangian L modifies, but the Hamiltonian H remains invariant.

6.4 Energy-momentum tensor field

Because a multi-time Hamiltonian is not conserved along the extremals of the associated Lagrangian, reasons from physics ask to introduce the *energy-momentum tensor field*

$$T^\alpha{}_\beta = x^i_\beta \frac{\partial L}{\partial x^i_\alpha} - L\delta^\alpha_\beta$$

This tensor field describes the density and flux of energy, and momentum in spacetime. If the Lagrangian L does not depend explicitly on the multi-parameter t, then the energy-momentum tensor field represents a conservation law in the sense that the divergence of the tensor field $T^\alpha{}_\beta$ is zero. *Find the energy-momentum tensor field for each of the previous Lagrangians.*

6.5 Interaction between vector fields

To describe the "interaction" between vector fields or 1-forms, when they are working together in a multitime physical or economical dynamical system, we recommend to use their generated distributions in the sense of the differential geometry. Of course, in this way we overpass the old idea of summing similar vector fields or similar 1-forms and after that of introducing the total vector field or total 1-form in a single-time dynamical system. We pass in fact to *multitime dynamical systems on distributions.*

6.6 Least squares Lagrangian associted to Maxwell PDEs

We recall some open problems from the book [21].

Let U be a domain of linear homogeneous isotropic media in the Riemannian manifold $(M = R^3, \delta_{ij})$. Maxwell's equations (coupled PDEs of first order)

div $D = \rho$, rot $H = J + \partial_t D$, div $B = 0$, rot $E = -\partial_t B$,

where ∂_t is the time derivative operator, together with the constitutive equations

$$B = \mu H,\ D = \epsilon E,$$

on $R \times U$, reflect the relations between the electromagnetic fields:

E $[V/m]$	electric field strength
H $[A/m]$	magnetic field strength
J $[A/m^2]$	electric current density
ϵ $[As/Vm]$	permitivity
μ $[Vs/Am]$	permeability
B $[T] = [Vs/m^2]$	magnetic induction
$-$ $-$	(magnetic flux density)
D $[C/m^2] = [As/m^2]$	electric displacement
$-$ $-$	(electric flux density)

Since div $B = 0$, the vector field B is source free, hence may be expressed as *rot* of some vector potential A, i.e., $B = \text{rot}\ A$. Then the electric field strength is $E = -\text{grad}\, V - \partial_t A$.

Find interpretations for the extremals of least squares Lagrangians of the type

$$L_1 = \frac{1}{2}||rot\, E + \partial_t B||^2 + \frac{1}{2}||rot\, H - J - \partial_t D||^2$$
$$+\frac{1}{2}(div\, D - \rho)^2 + \frac{1}{2}(div\, B)^2$$

which are not solutions of Maxwell equations. Can we derive, in this way, the Dirac theory of magnetic monopole?

Let us refer to Maxwell theory in terms of differential forms. In this sense, it is well-known that E, H are differential 1-forms, J, D, B are differential 2-forms, ρ is a differential 3-form, and star operator from $D = \epsilon * E$, $B = \mu * H$ is the Hodge operator. If d is the exterior derivative operator, and ∂_t is the time derivative operator, then the Maxwell's equations for static media are

$$dE = -\partial_t B,\ dH = J + \partial_t D,\ dD = \rho,\ dB = 0$$

(coupled PDEs of first order) on $R \times U$. *Find interpretations for the extremals of least squares Lagrangians*

$$L_2 = \frac{1}{2}||dE + \partial_t B||^2 + \frac{1}{2}||dH - J - \partial_t D||^2$$
$$+\frac{1}{2}||dD - \rho||^2 + \frac{1}{2}||dB||^2,$$

which are not solutions of Maxwell equations.

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