Convergence of the SUT_N approach of $\mathcal{C}_0$ semigroups for steady state with the total cross section and scattering kernel not depending on the spatial variable

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Abstract: In this paper we study the convergence of the SUT_N solution for the steady state transport equation in one dimensional case using the $\mathcal{C}_0$ semigroups with the total cross section and scattering kernel not depending on the spatial variable.

Keywords: transport equation,steady state,scattering kernel,cross section.

1. INTRODUCTION

The neutron transport equation which is an linear equation is a special case of the Boltzmann equation with wide applications in physics and engineering an models the transport of neutral particles in a scattering, absorbing, and emitting medium (14).

As is well known, the study of a given transport equation is a quite important and interesting in transport theory.

Among of these methods Various methods have been developed to investigate, and special attention has been given to the task of searching methods that generate accurate results to transport problems in the context of deterministic methods based on analytical procedures, for the multidimensional transport problems, one of the effective methods to treat linear transport equation is the spectral method (24), (23), (20) etc..., whose basic goals is to find exact solution for approximations of the transport equation, several approaches have been suggested. Among them, the method proposed by Chandrasekhar (14) solves analytically the discrete equations, (S_N equations), the spherical harmonics method (15) expands the angular flux in Legendre polynomials, the F_N method (18) transforms the transport equation into an integral equation. The integral transform technique like the Laplace, Fourier and Bessel also have been applied to solve the transport equation in semi-infinite domain (16), (17) the SGF method (3), (4) is a numerical nodal method that generates numerical solution for the S_N equations in slab geometry that is completely free of spatial truncation error. The LTS_N method (30; 26) solve analytically the S_N equations employing the Laplace Transform technique in the spatial variable (finite domain). Recently, following the idea encompassed by the LTS_N method, we have derived a generic method, prevailing the analyticity, for solving one-dimensional approximation that transform the transport equation into a set differential equations.

The version of this generic method are known as LTS_N (2), LTP_N (32), LTW_N (11), $LTCh_N$ (12), LTA_N (13), LTD_N (5).

Chebyshev spectral methods for radiative transfer problems are studied, e.g., by Kim and Ishimaru in (21) and by Kim and Moscoso in (22) and by Asadzadeh and Kadem in (1). For more detailed study on Chebyshev spectral method and also approximations by the spectral methods we refer the reader to monographs by Boyd (8) and Bernardi and Maday (7).

The purpose of this paper is to present the convergence of the SUT_N solution for the one dimensional transport equation, using Chebyshev polynomials (25) combined with the Sumudu transform. The approach is based on expansion of the angular flux in a truncated series of Chebyshev polynomials in the angular variable. By replacing this development in the transport equation, this which will result a first-order linear differential system is solved for the spatial function coefficients by application of the Sumudu transform technique (6) after that we use the $\mathcal{C}_0$ semigroup approach in order to study the convergence.

2. ANALYSIS

Let us consider the following mono-energetic $3-D$ transport equation:

$$\underline{\Omega}.\underline{\nabla}(\underline{r},\underline{\Omega}) + \sigma_t \Psi(\underline{r},\underline{\Omega}) = \int_{4\pi} \sigma_s(\underline{\Omega},\underline{\Omega'})\Psi(\underline{r},\underline{\Omega'})d\Omega' + \frac{1}{4\pi}Q(\underline{r}) \quad (1)$$

where

$$\underline{r} = (x, y, z) = spatial variable, \quad (2)$$

$$\underline{\Omega} = (\eta, \xi) = angular variable, \quad (3)$$

and

$$\sigma_s(\mu_0) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} \sigma_{sk} P_k(\mu_0) = differential\ scattering cross\ section, \quad (4)$$

with $\mu_0 = \underline{\Omega}.\underline{\Omega'}$ and $P_k =$ the k^{th} Legendre polynomial.

3. PLANAR GEOMETRY

We consider a planar-geometry problem with spatial variation only in the $x-$direction:

$$Q(\underline{r}) = q(x), \quad (5)$$

$$\Psi(\underline{r}, \underline{\Omega}) = \frac{1}{2\pi} \Psi(x, \mu). \quad (6)$$

Eq. (1) simplifies to

$$\mu \frac{\partial \Psi}{\partial x}(x, \mu) + \sigma_t \Psi(x, \mu) = \int_{-1}^{1} \sigma_s(\mu, \mu') \Psi(x, \mu') d\mu' + \frac{q(x)}{2}, (7)$$

with

$$\sigma_s(\mu, \mu') = \sum_{k=0}^{\infty} \frac{2k+1}{2} \sigma_{sk} P_k(\mu) P_k(\mu'). \quad (8)$$

So we consider Eq. (7) with $0 \le x \le a$ and $-1 \le \mu \le 1$, and subject to the boundary conditions

$$\Psi(x = 0, -\mu) = f(\mu), \quad (9)$$

and

$$\Psi(x = a, \mu) = 0, \quad (10)$$

where $f(\mu)$ is the prescribed incident flux at $x = 0$; $\Psi(x, \mu)$ is the angular flux in the μ direction; σ_t is the total cross section; σ_{sl}, with $l = 0, 1, ..., L$ are the components of the differential scattering cross section, and $P_k(\mu)$ are the Legendre polynomials of degree k.

Here we consider the base space is, $E = L^1([0, a] \times [-1, 1])$.

Theorem 3.1. *Consider the integro-differential equation* (7) *subject to the boundary conditions* (9) *and* (10)*, then the function* $\Psi(x, \mu)$ *satisfy the follow first-order linear differential equation system for the spatial component* $g_n(x)$

$$\sum_{n=0}^{N} \alpha^1_{n,m} g'_n(x) + \frac{\sigma_t \pi}{2 - \delta_{m,0}} g_m(x) = \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{m,l} \sum_{n=0}^{N} \alpha^3_{n,l} g_n(x) + \frac{q(x)}{2} \quad (11)$$

where

$$\alpha^1_{n,m} := \int_{-1}^{1} \mu T_n(\mu) \frac{T_m(\mu)}{\sqrt{1-\mu^2}} d\mu, \quad (12)$$

$$\alpha^2_{n,l} := \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu, \quad (13)$$

$$\alpha^3_{n,l} := \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu, \quad (14)$$

and $g_m(x)$ *are the coefficients of the expansion of the* $\Psi(x, \mu)$.

To prepare for the proof of the Theorem (3.1) we need the following result

Proposition 3.2. *Let*

$$T_{n+1}(x) - 2x T_n(x) + T_{n-1}(x) = 0 \quad (15)$$

and

$$P_{l+1}(x) = 2x P_l(x) - P_{l-1}(x) - [x P_l(x) - P_{l-1}(x)] / (l+1) (16)$$

the recurrence relations for the Chebyshev and the Legendre polynomials respectively we have for $l > 2$ *and* $k = 2, 3$

$$\alpha^k_{n,l+1} := \frac{2l+1}{2l+2} \left[\alpha^k_{n+1,l} + \alpha^k_{n-1,l}\right] - \frac{l}{l+1} \alpha^k_{n,j-1} \quad (17)$$

Hence, in particular for $l = 0$ *and* 1 *the coefficients* $\alpha^2_{n,l}$ *and* $\alpha^3_{n,l}$ *assume the values*

$$\alpha^2_{n,l} = \begin{cases} 0 & if n + l odd, \\ \frac{2}{(1+l)^2 - n^2} & if n + l even, \end{cases} \quad (18)$$

and

$$\alpha^3_{n,l} = \frac{\pi \delta_{n,l}}{2 - \delta_{l,0}}. \quad (19)$$

Proof

It easy to see that

$$\alpha^1_{n,m} = \frac{\pi \delta_{|n-m|}}{2(2 - \delta_{n+m,1})}. \quad (20)$$

For $k = 2$ by the multiplication of the Chebyshev and the Legendre recurrence formulas we have

$$\frac{2l+1}{2l+2} \left[P_l(\mu) T_{n+1}(\mu) + P_l(\mu) T_{n-1}(\mu)\right] - \frac{l}{2\mu(l+1)} P_{l-1}(\mu) \left[T_{n+1}(\mu) + T_{n-1}(\mu)\right]. \quad (21)$$

Since it is known that

$$T_{n+1}(\mu) + T_{n-1}(\mu) = 2\mu T_n(\mu) \quad (22)$$

after doing some algebraic manipulations and integrating over $\mu \in [-1, 1]$ on the resulting equation we get

$$\alpha^2_{n,l+1} = \frac{2l+1}{2l+2} \left[\alpha^2_{n+1,l} + \alpha^2_{n-1,l}\right] - \frac{l}{l+1} \alpha^2_{n,j-1}. \quad (23)$$

The case $k = 3$ is treated similarly but in this case we multiply the resulting expression by $\frac{1}{\sqrt{1-\mu^2}}$ and integrate over $\mu \in [-1, 1]$ we get the desired result.

Proof of Theorem 3.1

Expanding the angular flux in the μ variable in terms of the Chebyshev polynomials (25) leads to

$$\Psi(x, \mu) = \sum_{n=0}^{N} \frac{g_n(x) T_n(\mu)}{\sqrt{1-\mu^2}} \quad (24)$$

with $N = 0, 2, 4, ...$, where the expansions coefficients $g_n(x)$ should be determined.

Here $T_n(\mu)$ are the well known Chebyshev polynomials of order n which are orthogonal in the interval $[-1, 1]$ with respect to the weight $w(t) = 1/\sqrt{1-t^2}$.

After replacing Eq. (24) into Eq. (7) it turns out

$$\sum_{n=0}^{N} \{\mu g'_n(x) + \sigma_t g_n(x)\} \frac{T_n(\mu)}{\sqrt{1-\mu^2}} = \tag{25}$$

$$\sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} P_l(\mu) \sum_{n=0}^{N} g_n(x) \int_{-1}^{1} P_l(\mu') \frac{T_n(\mu')}{\sqrt{1-\mu'^2}} d\mu' + \frac{q(x)}{2}. \tag{26}$$

Using the orthogonality of the Chebyshev polynomials, multiply the Eq. (26) by $T_m(\mu)$,considering $m = 0, 1, ..., N$, and integrated in the μ variable in the interval $[-1, 1]$. Thus, we get the following first-order linear differential equation system for the spatial component $g_n(x)$

$$\sum_{n=0}^{N} \alpha^1_{n,m} g'_n(x) + \frac{\sigma_t \pi}{2-\delta_{m,0}} g_m(x)$$
$$= \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{m,l} \sum_{n=0}^{N} \alpha^3_{n,l} g_n(x) + \frac{q(x)}{2}, \tag{27}$$

where

$$\alpha^1_{n,m} = \int_{-1}^{1} \mu T_n(\mu) \frac{T_m(\mu)}{\sqrt{1-\mu^2}} d\mu, \tag{28}$$

$$\alpha^2_{n,l} = \int_{-1}^{1} T_n(\mu) P_l(\mu) d\mu, \tag{29}$$

$$\alpha^3_{n,l} = \int_{-1}^{1} \frac{T_n(\mu) P_l(\mu)}{\sqrt{1-\mu^2}} d\mu, \tag{30}$$

with $\delta_{n,m}$ denoting the delta of Kronecker. Here the coefficients $\alpha^2_{n,l}$ and $\alpha^3_{n,l}$ are evaluated by the multiplication of the Chebyshev and Legendre recurrence formulas and integration of the resulting equation (See proposition 3.2).

We now obtain the following first-order linear differential equation system (27) with the corresponding boundary conditions, writing in matrix form

$$\left\{ \begin{array}{c} A.\frac{dg}{dx}(x) + Bg(x) = C(x) \\ g(0) = f_n \\ g(a) = 0 \end{array} \right\} \tag{31}$$

where $g(x) =$ Col. $[g_0(x), g_1(x), ..., g_N(x)]$ and A and B are squared matrices of order $N+1$ with the components

$$(A)_{i,j} = \alpha^1_{i-1,j-1}, \tag{32}$$

$$(B)_{i,j} = \frac{\pi\sigma_t}{2-\delta_{1,j}} \delta_{i,j} - \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{i-1,l} \sum_{n=0}^{N} \alpha^3_{j-1,l} \tag{33}$$

and

$$C(x) = \frac{q(x)}{2} = Col. [C_0(x), C_1(x), ..., C_N(x)]. \tag{34}$$

The S_n discrete method consists in selecting a finite number of discrete directions μ_{k_i} in the interval $[-1, 1]$, and we take the discrete ordinate of the equation (31)

$$\left\{ \begin{array}{c} A_{i,j}.\frac{dg}{dx}(x) + B_{i,j} g(x) = C(x) \\ g(0) = f_k \\ g(a) = 0 \end{array} \right. \tag{35}$$

We define the following arrays

$$A_{i,j} = \int_{-1}^{1} \mu_k T_n(\mu_k) \frac{T_m(\mu_k)}{\sqrt{1-\mu_k^2}} d\mu_k, \tag{36}$$

$$B_{i,j} = \frac{\pi\sigma_t}{2-\delta_{1,j}} \delta_{i,j}$$
$$- \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \int_{-1}^{1} T_n(\mu_k) P_l(\mu_k) d\mu_k \sum_{n=0}^{N} \int_{-1}^{1} \frac{T_n(\mu_k) P_l(\mu_k)}{\sqrt{1-\mu_k^2}} d\mu_k \tag{37}$$

and we notice that this equation has the well known solution (28)

$$g(x) = e^{-A_k^{-1} B_k x} g(0) + \int_{0}^{x} e^{-A_k^{-1} B_k (x-\xi)} C(\xi) d\xi, \tag{38}$$

with

$$A_k^{-1} B_k = \left(\alpha^1_{i-1,j-1}\right)^{-1} \frac{\pi\sigma_t}{2-\delta_{1,j}} \delta_{i,j}$$
$$- \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{i-1,l} \sum_{n=0}^{N} \alpha^3_{j-1,l}. \tag{39}$$

The space E_n approximate E, and the SUT_n transport operator $\mathcal{K}_n = A_n^{-1} B_n$ approximate the original transport operator $\mathcal{K}$ given by the equation (1), so the solution of equation (1) is given in the operator form

$$\left\{ \begin{array}{c} \Psi(x,\mu) = e^{-\frac{1}{\mu}(\sigma_t I - \mathcal{J})x} \Psi(0,\mu) \\ with \mathcal{J}\Psi(x,\mu) = \int_{-1}^{1} \sigma_s(\mu,\mu') \Psi(x,\mu') d\mu' \end{array} \right. \tag{40}$$

We know that the semigroups generated by the operator $A_k^{-1} B_k$ satisfy the stability condition, i.e.

$$\| e^{-A_k^{-1} B_k (x-\xi)} \| \leq N e^{\omega(x-\xi)} \tag{41}$$

in fact

$$\| A_k^{-1} B_k \| \leq \| N^{-1} \| \; \| \left(\alpha^1_{i-1,j-1}\right)^{-1} \frac{\pi\sigma_t}{2-\delta_{1,j}} \delta_{i,j}$$
$$- \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{i-1,l} \sum_{n=0}^{N} \alpha^3_{j-1,l} \| \tag{42}$$

$$\leq \max_j \left([\alpha^1_{i-1,j-1}\right)^{-1} \frac{(\pi\sigma_t)\delta_{i,j}}{2-\delta_{1,j}} - \sum_{l=0}^{L} \frac{2l+1}{2} \sigma_{sl} \alpha^2_{i-1,l} \sum_{n=0}^{N} \alpha^3_{j-1,l}]. \tag{43}$$

The approximation theorem of $\mathcal{C}_0$ semigroups (19) asserts that $e^{-A_k^{-1} B_k (x-\xi)}$ converges to the $\mathcal{C}_0$ semigroups generated by $\mathcal{K}$ for x in compact subsets of R^+. In this way the convergence is obtained.

4. CONCLUSION

An adaptation of the method for the convergence of the spectral solution within the framework of the analytical solution to study and prove the existence and uniqueness of the solution by using the discrete ordinates method is essential for the linear transport in specially case steady state with the total cross section and scattering kernel not depending on the spatial variable. Here we use C_0 *semigroup* approach for considering that the total and scattering kernel are both independent with respect to the spatial variables.

We can assert that the Chebyshev spectral method combined with Sumudu transform should be general enough to consider higher spatial dimensions in a way similar to that presented in this paper. In preparation for these problems, we are currently investigating the effectiveness of spectral methods combined with Sumudu transform in solving the linear system of differential equation analytically and to prove the convergence we shall use the approximation theorem.

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