Hamiltonian Structure of the Magnetic Field Lines

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Abstract: In this paper a geometrical interpretation is given to the divergenceless equations particularly to the magnetic field lines in confined plasmas. By using the geometrical concepts they are formulated in Hamiltonian and Nambu formalisms. By writing these equations in the language of the geometrical objects and by using the geometrical concepts we could simplify the formalization. As Hamiltonian formalization has advantage of using the mechanical tools, the geometrizing simplifies the calculation and the formalization and it reveals several deep insights.

Keywords: integrable and non-integrable systems, forms, Lie derivative, exterior derivative, magnetic surfaces

1. INTRODUCTION

Tokamaks are confined plasma machines that their stabilities are an important problem in which the structure of magnetic field lines plays an fundamental role(1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12). In addition the plasma confinement in tokamak can be improved by creating a chaotic fields line region which decreases the plasma-wall interaction. This chaotic region is created by suitably designed resonant magnetic windings which produce the overlapping of two or more chains of magnetic islands and, consequently, the magnetic surfaces of destruction. Magnetic fields lines in general are orbits of hamiltonian systems of one and a half degree. Expressing these equation in hamiltonian form has advantage of using powerful methods of hamiltonian mechanics like perturbation theory and KAM and adiabatic invariance and etc(12). The magnetic field line studies indicate the non-integrable divergenceless flows in R^3 may have dense, and space filling orbits and may show extreme sensibility to initial condition, leading to a loss of predictability, even though the system is deterministic. When a field is represented by means of a magnetic field line Hamiltonian the field lines can be identified with phase space trajectories produced by the Hamiltonian. Mathematically, the issue of magnetic surface reduces to a problem in Hamiltonian mechanics meaning that the whole of mechanics methods is appellable to this problem for example perturbation theory and others. The Hamiltonian mechanics itself can be formulated in a geometric way that in this paper we use this formalizations for magnetic field lines. This paper is organized in four sections: In the first section a vector analysis of magnetic field lines is outlined and the hamiltonian structure of these equations is derived . This section is divided to subsections. Subsection 2 is devoted to two integrable condition of equations. One of them is analytical way and another is numerical. In subsection 3 we discuss the formalization in terms of forms. This method is used to show the Hamiltonian structure of the magnetic field lines. In the next subsection the Lie derivative of forms is applied to show this structure, and finally the Poisson bracket method is investigated. In Section 4 we discuss the Nambu structure of the magnetic fields because this formalization is suitable for describing non-integrable systems. In Section 5 we present our conclusion.

2. HAMILTONIAN FORMULATION

2.1 Vector analysis

The equations of motion of magnetic field lines are given by dynamical system of equation,

$$\dot{\mathbf{x}} = \mathbf{B}, \tag{1}$$

where $\nabla \cdot \mathbf{B} = 0$. When $\mathbf{B}$ is a nonlinear function of $\mathbf{x}$ dynamical systems the theory implies that the solutions may exhibit deterministic chaos (e.g. show extreme sensitivity to the initial condition). Integrable divergence less flows in R^3 have only periodic and homo clinic or hetero clinic orbits, whose behavior is regular and dose not lose predictability over time. We treat two kind of magnetic fields where the magnetic field lines is restricted to a flux surface and another cover ergodically a finite volume. The second field lines are not so much different from the first(near integrable systems). All of the magnetic fields equations have Hamiltonian structure form. Since $\nabla \cdot \mathbf{B} = 0$

, $\mathbf{B} = \nabla \times \mathbf{A}$ in an arbitrary orthogonal system, ρ, θ, ζ, without assumption of existence of magnetic surfaces,

$$\mathbf{A} = A_\rho \nabla\rho + A_\theta \nabla\theta + A_\zeta \nabla\zeta, \tag{2}$$

using $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\mathbf{A} + \nabla G)$ so

$$\mathbf{A} = \Psi\nabla\theta + \Phi\nabla\zeta + \nabla G, \tag{3}$$

where $\frac{\partial G}{\partial \theta} = A_\theta - \Psi$, $\frac{\partial G}{\partial \zeta} = A_\zeta - \Phi$, $\frac{\partial G}{\partial \rho} = A_\rho$ and $G = 0$ at $\rho = 0$. Upon taking the curl, we obtain

$$\mathbf{B} = \nabla\Psi \times \nabla\theta + \nabla\Phi \times \nabla\zeta. \tag{4}$$

(4) contrast with (1) is called canonical representation. In proper magnetic surface case (intergrade case) the ψ_p and ψ_t is function of ρ. In other words we have $\psi_p(\psi_t)$. But in general case Ψ and Φ are dependent on all variables, ρ, θ, ζ , $\mathbf{B} \cdot \nabla\Phi \neq 0$, $\mathbf{B} \cdot \nabla\Psi \neq 0$, and

$$\Phi(\Psi, \theta, \zeta). \tag{5}$$

Using $\mathbf{B} \times d\mathbf{l} = 0$, $d\mathbf{l}$ is tangent to the direction of magnetic line, the Hamiltonian form equations is obtained (1),

$$\frac{\partial \Psi}{\partial \zeta} = \frac{\partial \Phi}{\partial \theta}, \tag{6}$$

$$\frac{d\theta}{d\zeta} = -\frac{\partial \Phi}{\partial \Psi}. \tag{7}$$

Considering (4) we obtain

$$\frac{d\Psi}{ds} = \mathbf{B} \cdot \nabla\Psi = \nabla\Phi \times \nabla\zeta \cdot \nabla\Psi, \tag{8}$$

$$\frac{d\theta}{ds} = \mathbf{B} \cdot \nabla\theta = \nabla\Phi \times \nabla\zeta \cdot \nabla\theta, \tag{9}$$

$$\frac{d\zeta}{ds} = \mathbf{B} \cdot \nabla\zeta = \nabla\Psi \times \nabla\theta \cdot \nabla\zeta. \tag{10}$$

Poisson bracket is define as follows

$$\{F, G\} = \nabla\zeta \cdot \nabla F \times \nabla G = \frac{1}{J}\{\frac{\partial f}{\partial \Psi}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \Psi}\}, \tag{11}$$

which satisfies antisymmetric condition

$$\{f, g\} = -\{g, f\} \tag{12}$$

and Jacobi identity

$$\{f, \{g, h\}\} + \{f, \{g, h\}\} + \{f, \{g, h\}\} = 0. \tag{13}$$

The manifold for which the Poisson bracket is defined on it is called Poisson manifold. ζ coordinate is the Casimir invariant of the system since

$$\{\zeta, f\} = \nabla\zeta \cdot \nabla\zeta \times \nabla f = 0. \tag{14}$$

The equations of motion in Poisson bracket notation is given by

$$\dot{\eta} = \{\eta, H\}, \tag{15}$$

where η is replaced with Ψ, θ, ζ, then the following equations are obtained

$$\frac{d\Psi}{ds} = \frac{1}{J}\frac{\partial \Phi}{d\theta} \tag{16}$$

$$\frac{d\theta}{ds} = \frac{1}{J}\frac{\partial \Phi}{d\Psi} \tag{17}$$

$$\frac{d\zeta}{ds} = 0. \tag{18}$$

Taking $J = h(\Psi)h'(\theta)$, J can be eliminated by taking new variables. For example let us consider the canonical bracket of

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}. \tag{19}$$

For the following coordinate transformation

$$x = r\cos\theta \tag{20}$$

$$y = r\sin\theta,$$

we have

$$\{f, g\} = \frac{1}{r}(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial r}). \tag{21}$$

Let J is the Jacobian of (20) and choosing $J = \frac{1}{2}r^2$ we obtain

$$\{f, g\} = \frac{\partial f}{\partial J}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial J}, \tag{22}$$

which is in the standard form. From (10) the condition of Jacobian is obtained

$$B_\zeta = J. \tag{23}$$

Example Consider the magnetic field

$$\mathbf{B} = (0, B_\theta(r), B_0). \tag{24}$$

Choosing the general method and coordinate system r, θ, ϕ Hamiltonian Φ and conjugate momentum Ψ can be found as follows

$$\mathbf{B} \cdot \nabla\theta = \nabla\Phi \times \nabla\phi \cdot \nabla\theta, \tag{25}$$

and

$$\mathbf{B} \cdot \nabla\theta = \frac{\partial \Phi}{\partial r}\nabla r \times \nabla\phi \cdot \nabla\theta \tag{26}$$

then

$$\int R_0 r B_\theta dr = \int \frac{\partial \Phi}{\partial r} dr \tag{27}$$

or

$$\int \frac{B_\theta}{B_\phi} d\Psi = \int \frac{\partial \Phi}{\partial \Psi} d\Psi. \tag{28}$$

And an equation for Ψ is obtained,

$$B \cdot \nabla\phi = \frac{\partial \Psi}{\partial r}\nabla r \times \nabla\phi \cdot \nabla\theta, \tag{29}$$

$$B_\phi = \frac{\partial \Psi}{\partial r}\nabla r \times \nabla\phi \cdot \nabla\theta, \tag{30}$$

$$\int r R_0 B_\phi dr = \int \frac{\partial \Psi}{\partial r} dr. \tag{31}$$

It is noticeable that there is no assumption about integrability of system. For this sake the general method is used which is applicable for integrable and non-integrable systems.

It is interesting to do the above calculations in Cartesian coordinate system, then consider

$$\mathbf{A} = a_x\nabla x + a_y\nabla y + a_z\nabla z, \tag{32}$$

choosing the direction of $\mathbf{A}$ such that $a_x = 0$, then

$$\dot{x} = \frac{\partial a_y}{\partial z} - \frac{\partial a_z}{\partial y} \tag{33}$$

$$\dot{y} = \frac{\partial a_z}{\partial x} \tag{34}$$

$$\dot{z} = \frac{\partial a_z}{\partial y}, \tag{35}$$

so

$$\nabla \times \mathbf{A} = \nabla a_y \times \nabla y + \nabla a_z \times \nabla z \tag{36}$$

changing the coordinate system , $a_z(a_y, y, z)$, we obtain

$$\frac{da_y}{dz} = \frac{\partial a_x}{\partial y}, \tag{37}$$

$$\frac{dy}{dz} = -\frac{\partial a_x}{\partial a_y} \tag{38}$$

which are in Hamiltonian structure.

2.2 Integrable systems: theoretical and numerical method

Consider the dynamical systems which can be written in the following form

$$\dot{\mathbf{x}} = \nabla u \times \nabla v. \tag{39}$$

It can be defined a Poisson bracket for these systems as follows

$$\{F, G\} = \nabla u \cdot \nabla F \times \nabla G. \tag{40}$$

u is the Casimir function because of

$$\{u, G\} = 0. \tag{41}$$

The equation of motion is written as follows

$$\dot{\mathbf{x}} = \nabla u \cdot \nabla \mathbf{x} \times \nabla v. \tag{42}$$

The systems like (39) are integrable systems. It can be proved that the well-behaved magnetic field lines systems can be written in the following form

$$\mathbf{B} = \nabla \psi_t \times \nabla(\theta_t - \iota \zeta_t), \tag{43}$$

where $\iota = \frac{d\psi_p}{d\psi_t}$. Then, the well-behaved magnetic fields are integrable systems.

Example Let us Consider the equations

$$\dot{x} = -y \qquad \dot{y} = x, \tag{44}$$

which can be written as

$$\dot{\mathbf{x}} = \nabla H \times \nabla z. \tag{45}$$

z component is Casimir variable and H is the Hamiltonian of system. Then (44) is integrable.

Example Let us consider the equations,

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\theta} &= \omega_1 \\ \dot{\phi} &= \omega_2, \end{aligned} \tag{46}$$

which can be written in the following form

$$\dot{\mathbf{r}} = \nabla H(r) \times \nabla(\omega_1 \theta + \omega_2 \phi), \tag{47}$$

so this system is integrable. Numerically the integrable systems have regular cross section (Figure 1,2). All the rational trajectory solutions are periodic and enough irrational solutions densely fill the magnetic surfaces.

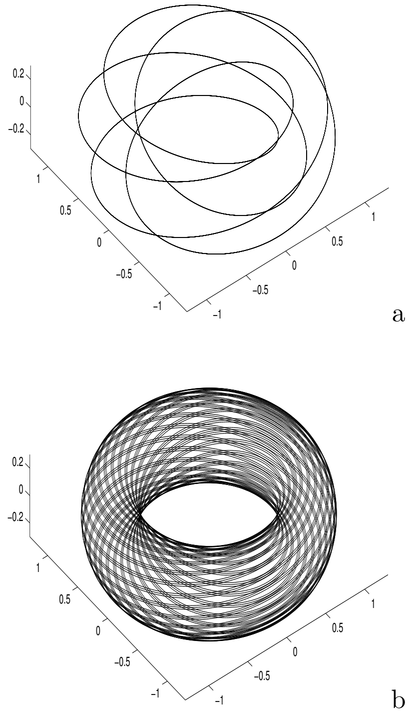

Fig. 1. In Cartesian system (x(r,θ, ϕ),y(r,θ, ϕ),z(r,θ, ϕ))a)periodic solution b)enough irrational solutions fills densely the surface.

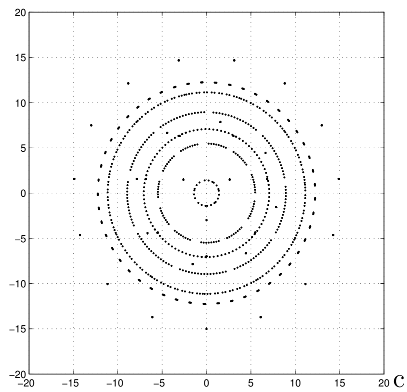

Fig. 2. (y,z)-plane with cross-section x=0 c)cross-section of periodic and irrational solutions.

2.3 Formalization in terms of forms

Consider the following form

$$\alpha = B_x dydz + B_y dxdz + B_z dxdy. \tag{48}$$

For the divergenceless forms,

$$d\alpha = (\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z})dxdydz = 0, \tag{49}$$

we have $\alpha = d\nu$. This is the equivalence of vector analysis which says that if $\nabla \cdot \mathbf{B} = 0$, then $\mathbf{B} = \nabla \times \mathbf{A}$.

The Hamiltonian equations can be obtained from the following relation

$$(dp + \frac{\partial H}{\partial q}dt)(dq - \frac{\partial H}{\partial p}dt) = 0, \tag{50}$$

then

$$dpdq + \frac{\partial H}{\partial q}dtdq - \frac{\partial H}{\partial p}dpdt = 0. \tag{51}$$

Equating (48)and (51) we obtain

$$B_z = J, \tag{52}$$

where J is the Jacobi of transformation from (p, q) to (x, y), and

$$\frac{\partial x}{\partial q} = \frac{1}{J}\frac{\partial p}{\partial y} \qquad \frac{\partial x}{\partial p} = -\frac{1}{J}\frac{\partial q}{\partial y}, \tag{53}$$

$$\frac{\partial y}{\partial p} = \frac{1}{J}\frac{\partial q}{\partial x} \qquad \frac{\partial y}{\partial q} = -\frac{1}{J}\frac{\partial p}{\partial x}. \tag{54}$$

Using the identities

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy \tag{55}$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy \tag{56}$$

and (53),(54) we can calculate H, p, q and in addition we can define the Poisson bracket as follows

$$\{H, f\} = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}. \tag{57}$$

2.4 Hamiltonian vector fields

Along Hamiltonian vector fields, U, the Lie derivative of $\omega = dp \wedge dq$ is zero or

$$L_U\omega = 0. \tag{58}$$

Changing the coordinate to non-canonical coordinate system then

$$\omega = dp \wedge dq = \frac{1}{J}dx \wedge dy. \tag{59}$$

Consider the desired equations

$$\frac{dx}{dz} = \frac{B_x}{B_z}, \tag{60}$$

$$\frac{dy}{dz} = \frac{B_y}{B_z}. \tag{61}$$

These equations having Hamiltonian structure they must satisfy the equation (58) along the following vector field

$$U = \frac{d}{dz} = \frac{dx}{dz}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}. \tag{62}$$

So, we have

$$\omega(U) = \frac{1}{J}\frac{dx}{dz}dx + \frac{1}{J}\frac{dy}{dz}dy. \tag{63}$$

From (4)

$$L_U(\omega) = d\omega = 0, \tag{64}$$

then

$$d\omega = \frac{\partial}{\partial x}(\frac{1}{J}\frac{B_x}{B_z})dx + \frac{\partial}{\partial y}(\frac{1}{J}\frac{B_y}{B_z})dy \tag{65}$$

and

$$d\omega = B \cdot \nabla \ln J - B \cdot \nabla \ln B_z = B \cdot \nabla \ln(\frac{J}{B_z}) = 0. \tag{66}$$

Finally, the condition of such a transformation is obtained

$$B_z = J. \tag{67}$$

2.5 Poisson Brackets, Hamiltonian dynamics and the magnetic field equations

Consider the definition of canonical Poisson bracket

$$\{f, g\}_{p,q} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}, \tag{68}$$

by transforming to coordinate system (u^i) we have

$$\{f, g\}_{u_1,u_2} = \frac{1}{J}(\frac{\partial f}{\partial u^i}\frac{\partial g}{\partial u^j} - \frac{\partial g}{\partial u^i}\frac{\partial f}{\partial u^j}). \tag{69}$$

From mechanics, a dynamical system is Hamiltonian if and only if the time derivative of PB obeys the following rule(3)

$$\frac{d}{dt}\{f, g\} = \{\dot{f}, g\} + \{f, \dot{g}\}. \tag{70}$$

The equations ,(60),(61), being in Hamiltonian structure they must verify the following relation

$$\frac{d}{dz}\{x, y\} = 0 = \{\dot{x}, y\} + \{x, \dot{y}\} = \frac{\partial}{\partial x}(\frac{1}{J}\frac{B_x}{B_z}) - \frac{\partial}{\partial y}(\frac{1}{J}\frac{B_y}{B_z}), \tag{71}$$

and using free divergence the relation, $\nabla \cdot \mathbf{B} = 0$ we lead

$$B \cdot \nabla \ln(J) = B \cdot \nabla \ln(B_z), \tag{72}$$

as a result

$$J = B_z. \tag{73}$$

Then, if the equations(60),(61) are in Hamiltonian structure they must satisfy the condition (73).

3. NAMBU STRUCTURE OF THE MAGNETIC FIELD LINES

The dynamical system given by (1) is obviously non Hamiltonian(the dimension of the phase space is odd). But it is shown that the following volume form is preserved along the magnetic field lines then consider

$$\omega = dx \wedge dy \wedge dz \tag{74}$$

and,

$$\omega = dx \otimes dy \otimes dz - dx \otimes dz \otimes dy - dy \otimes dx \otimes dz + dy \otimes dz \otimes dx + dz \otimes dy \otimes dx - dz \otimes dx \otimes dy, \tag{75}$$

where $dx \wedge dy = dx \otimes dy - dy \otimes dx$. Consider vector field

$$\mathbf{B} = B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}, \tag{76}$$

from the Lie derivative identity

$$\mathrm{L}_{\mathbf{B}}(\omega) = d(\omega(\mathbf{B})) + \mathbf{B}d\omega \tag{77}$$

and since $d\omega = 0$ we have

$$\mathrm{L}_{\mathbf{B}}\omega = d(\omega(\mathbf{B})), \tag{78}$$

then

$$\mathrm{L}_{\mathbf{B}}\omega = (\mathrm{div}\ \mathbf{B})\omega. \tag{79}$$

Since div $\mathbf{B} = 0$ or any other divergence-less field,

$$L_{\mathbf{B}}\omega = 0. \tag{80}$$

From other side

$$\mathbf{B} = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}, \tag{81}$$

then

$$\omega(\mathbf{B}) = \frac{dx}{dt}dydz + \frac{dy}{dt}dxdz + \frac{dz}{dt}dxdz, \tag{82}$$

and

$$B_x dydz + B_y dxdz + B_z dxdy. \tag{83}$$

From Frobenius theorem equations (83) can be written as

$$B_x dydz + B_y dxdz + B_z dxdy = dh_1 \wedge dh_2. \tag{84}$$

The condition for the exitances of the function h_1, h_2 is

$$\Omega = Pdx + Qdy + Rdz \tag{85}$$

and

$$\begin{aligned} \Omega \wedge d\Omega = P(R_y - Q_z)dx + Q(P_z - R_x)dy \\ +R(Q_x - P_y)dz = 0, \end{aligned} \tag{86}$$

where

$$d\Omega = \mathbf{B}. \tag{87}$$

It is important to say that the Nambu method is suitable for describing the non-integrable systems.

4. CONCLUSION

In this article we use the geometrical methods for magnetic field lines. The geometric formulation of Nambu mechanics is applied to the magnetic field lines and it is found the condition for the existence of the Nambu functions for magnetic field lines. As it is known this formalism is suitable for non-integrable systems. A geometrical formulation has many advantage in dynamical systems for example enable one to treat some general issues such as integral invariants and canonical transformations in a simple way and also it gives one deep insights.

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