Weyl's Limit Point and Limit Circle Cases for a Dynamic Equation on Time Scales

Adil Huseynov

Department of Mathematics, Ankara University, 06100 Tandogan, Ankara, Turkey (e-mail: Adil.Huseynov@science.ankara.edu.tr)

Abstract: We show that analogues of classical concepts of the Weyl limit point and limit circle cases can be introduced and investigated for second order linear dynamic equations on time scales. Since dynamical equations on time scales unifies and extends continuous and discrete dynamical equations (i.e., differential and difference equations), in this way we establish more general theory of the limit point and limit circle cases.

Keywords: Time scale, delta and nabla derivatives, delta and nabla integrals, limit point case, limit circle case.

1. INTRODUCTION

The present paper deals with the second order linear dynamic equations (differential equations on time scales are called the dynamic equations) on the semi-unbounded time scale intervals of the form

$$-[p(t)y^{\nabla}(t)]^{\Delta}+q(t)y(t)=\lambda y(t), \quad t\in(a,\infty)_{\mathbb{T}}, \tag{1}$$

and develops for such equations an analogue of the classical Weyl limit point and limit circle theory given by him for the usual Sturm-Liouville equation

$$-[p(t)y'(t)]'+q(t)y(t)=\lambda y(t), \quad t\in(a,\infty)_{\mathbb{R}}, \tag{2}$$

in the first decade of the twentieth century, [12]. The limit point and limit circle theory plays an important role in the spectral analysis of differential equations on unbounded intervals (see [5, 11, 13]). For discrete analogues of Eq. (2) (for infinite Jacobi matrices) the concepts of the limit point and limit circle cases were introduced and investigated by Hellinger [8] (see also [1, Chapter 1]).

Our consideration of the problem in this paper for Eq. (1) on time scales allows to unify the known continuous and discrete cases (i.e., differential and difference equations) and extend them to more general context of time scales.

For a general introduction to the calculus and the dynamic equations on time scales we refer the reader to [3, 4, 9].

2. LIMIT POINT AND LIMIT CIRCLE

Let $\mathbb{T}$ be a time scale which is bounded from below and unbounded from above so that

$$\inf\mathbb{T}=a>-\infty \quad \text{and} \quad \sup\mathbb{T}=\infty.$$

By the closedness of $\mathbb{T}$ in $\mathbb{R}$ we have $a\in\mathbb{T}$. We will denote such $\mathbb{T}$ also as $[a,\infty)_{\mathbb{T}}$ and call it a *semi-infinite* (or *semi-unbounded*) *time scale interval.*

Consider the equation

$$-[p(t)y^{\nabla}(t)]^{\Delta}+q(t)y(t)=\lambda y(t), \quad t\in(a,\infty)_{\mathbb{T}}, \tag{3}$$

where $p(t)$ is a real-valued Δ-differentiable function on $[a,\infty)_{\mathbb{T}}$ with piecewise continuous Δ-derivative $p^{\Delta}(t)$, $p(t)\neq 0$ for all t, $q(t)$ is a real-valued piecewise continuous function on $[a,\infty)_{\mathbb{T}}$, λ is a complex parameter (spectral parameter).

We define the *quasi ∇-derivative* $y^{[\nabla]}(t)$ of y at t by

$$y^{[\nabla]}(t)=p(t)y^{\nabla}(t).$$

For any point $t_0\in[\sigma(a),\infty)_{\mathbb{T}}$ and any complex constants c_0, c_1 Eq. (3) has a unique solution y satisfying the initial conditions

$$y(t_0)=c_0, \quad y^{[\nabla]}(t_0)=c_1.$$

If $y_1,y_2:[a,\infty)_{\mathbb{T}}\to\mathbb{C}$ are two ∇-differentiable on $[\sigma(a),\infty)_{\mathbb{T}}$ functions, where σ denotes the forward jump operator in $\mathbb{T}$, then the Wronskian of y_1 and y_2 is defined for $t\in[\sigma(a),\infty)_{\mathbb{T}}$ by

$$\begin{aligned} W_t(y_1,y_2) &= y_1(t)y_2^{[\nabla]}(t)-y_1^{[\nabla]}(t)y_2(t) \\ &= p(t)[y_1(t)y_2^{\nabla}(t)-y_1^{\nabla}(t)y_2(t)]. \end{aligned} \tag{4}$$

The Wronskian of any two solutions of Eq. (3) is independent of t. Two solutions of Eq. (3) are linearly independent if and only if their Wronskian is nonzero. Eq. (3) has two linearly independent solutions and every solution of Eq. (3) is a linear combination of these solutions.

We say that y_1 and y_2 form a *fundamental set* (or a *fundamental system*) of solutions for Eq. (3) provided their Wronskian is nonzero.

Let us consider the nonhomogeneous equation

$$-[p(t)y^{\nabla}(t)]^{\Delta}+q(t)y(t)=h(t), \quad t\in(a,\infty)_{\mathbb{T}}, \tag{5}$$

where $h:\mathbb{T}\to\mathbb{C}$ is a piecewise continuous function. If y_1 and y_2 form a fundamental set of solutions of the homogeneous equation (3) and $\omega=W_t(y_1,y_2)$, then the general solution of the corresponding nonhomogeneous equation (5) is given by

$$y(t)=c_1y_1(t)+c_2y_2(t)$$
$$+\frac{1}{\omega}\int_{t_0}^{t}[y_1(t)y_2(s)-y_1(s)y_2(t)]h(s)\Delta s, \tag{6}$$

where t_0 is a fixed point in $\mathbb{T}$, c_1 and c_2 are arbitrary constants. Formula (6) is called the *variation of constants formula.*

Let L denote the linear operator defined by

$$Lx = -(px^{\nabla})^{\Delta} + qx.$$

Lemma 1. If $Lx(t,\lambda) = \lambda x(t,\lambda)$ and $Ly(t,\lambda') = \lambda' y(t,\lambda')$, then for any $b \in (a,\infty)_{\mathbb{T}}$,

$$(\lambda' - \lambda)\int_{\sigma(a)}^{b} xy\Delta t = W_{\sigma(a)}(x,y) - W_b(x,y), \qquad (7)$$

where $W_t(x,y)$ is the Wronskian of x and y defined by (4).

Proof. We have, using the integration by parts formula

$$\int_c^d f^{\Delta}(t)g(t)\Delta t = f(t)g(t)\mid_c^d - \int_c^d f(t)g^{\nabla}(t)\nabla t$$

established in [7], that

$$\begin{aligned}(\lambda' - \lambda)\int_{\sigma(a)}^{b} xy\Delta t &= \int_{\sigma(a)}^{b}(xLy - yLx)\Delta t \\ &= -\int_{\sigma(a)}^{b}[x(py^{\nabla})^{\Delta} - y(px^{\nabla})^{\Delta}]\Delta t \\ &= -xpy^{\nabla}\mid_{\sigma(a)}^{b} + \int_{\sigma(a)}^{b} x^{\nabla}py^{\nabla}\nabla t \\ &\quad + ypx^{\nabla}\mid_{\sigma(a)}^{b} - \int_{\sigma(a)}^{b} y^{\nabla}px^{\nabla}\nabla t \\ &= -p(xy^{\nabla} - x^{\nabla}y)\mid_{\sigma(a)}^{b} \\ &= W_{\sigma(a)}(x,y) - W_b(x,y).\end{aligned}$$

The proof is complete.

Corollary 2. If, in particular, $\lambda = u + iv$, $\underline{\lambda' = \overline{\lambda}} = u - iv$ $(u, v \in \mathbb{R})$, then we can take $y(t,\lambda') = \overline{x(t,\lambda)}$ and (7) yields

$$2v\int_{\sigma(a)}^{b} |x(t,\lambda)|^2\,\Delta t = i\{W_{\sigma(a)}(x,\overline{x}) - W_b(x,\overline{x})\}. \qquad (8)$$

Let $\varphi(t,\lambda)$, $\theta(t,\lambda)$ be two solutions of Eq. (3) satisfying the initial conditions

$$\varphi(\sigma(a),\lambda) = \sin\alpha, \quad \varphi^{[\nabla]}(\sigma(a),\lambda) = -\cos\alpha, \qquad (9)$$

$$\theta(\sigma(a),\lambda) = \cos\alpha, \quad \theta^{[\nabla]}(\sigma(a),\lambda) = \sin\alpha, \qquad (10)$$

where $0 \le \alpha < \pi$. (Note that the ∇-nabla derivative is not defined at a if a is right-scattered). Then, since the Wronskian of any two solutions of (3) does not depend on t, we get

$$W_t(\varphi,\theta) = W_{\sigma(a)}(\varphi,\theta) = \sin^2\alpha + \cos^2\alpha = 1.$$

Then φ, θ are linearly independent solutions of (3), φ, $\varphi^{[\nabla]}$, θ, $\theta^{[\nabla]}$ are entire functions of λ and continuous in (t,λ). These solutions are real for real λ. Every solution y of (3) except for φ is, up to a constant multiple, of the form

$$y = \theta + l\varphi \qquad (11)$$

for some number l which will depend on λ.

Take now a point $b \in (a,\infty)_{\mathbb{T}}$ and consider the boundary condition

$$y(b)\cos\beta + y^{[\nabla]}(b)\sin\beta = 0 \quad (0 \le \beta < \pi) \qquad (12)$$

and ask what must l be like in order that the solution y, (11), satisfy (12). If denote the corresponding value of l by $l_b(\lambda)$, then we find that

$$l_b(\lambda) = -\frac{\theta(b,\lambda)\cot\beta + \theta^{[\nabla]}(b,\lambda)}{\varphi(b,\lambda)\cot\beta + \varphi^{[\nabla]}(b,\lambda)}.$$

Let us take any complex number z and introduce the function

$$l = l_b(\lambda,z) = -\frac{\theta(b,\lambda)z + \theta^{[\nabla]}(b,\lambda)}{\varphi(b,\lambda)z + \varphi^{[\nabla]}(b,\lambda)}. \qquad (13)$$

If b and λ are fixed, and z varies, (13) may be written as

$$l = \frac{Az + B}{Cz + D}, \qquad (14)$$

where $A = -\theta(b,\lambda)$, $B = -\theta^{[\nabla]}(b,\lambda)$, $C = \varphi(b,\lambda)$, $D = \varphi^{[\nabla]}(b,\lambda)$. Since

$$AD - BC = W_b(\varphi,\theta) = 1 \neq 0,$$

the linear-fractional transformation (14) is one-to-one conformal mapping which transforms circles into circles; straight lines being considered as circles with infinite radii. Besides, applying (8) to the solution $\varphi(t,\lambda)$ and taking into account (9) by virtue of which $W_{\sigma(a)}(\varphi,\overline{\varphi}) = 0$, we have

$$\begin{aligned}&2v\int_{\sigma(a)}^{b} |\varphi(t,\lambda)|^2\,\Delta t \\ &= -i\varphi(b,\lambda)\overline{\varphi}^{[\nabla]}(b,\lambda) + i\varphi^{[\nabla]}(b,\lambda)\overline{\varphi}(b,\lambda)\end{aligned}$$

which implies that $\varphi(b,\lambda) \neq 0$ and $\varphi^{[\nabla]}(b,\lambda) \neq 0$ if $Im\lambda = v \neq 0$. Therefore, if $Im\lambda = v \neq 0$, then $l_b(\lambda,z)$ varies on a circle $C_b(\lambda)$, with a finite radius, in the l-plane, as z varies over the real axis of the z-plane.

The center and the radius of the circle $C_b(\lambda)$ will be defined as follows. The center of the circle is the symmetric point of the point at infinity with respect to the circle. Thus if we set

$$l_b(\lambda,z') = \infty \quad \text{and}$$
$$l_b(\lambda,z'') = \text{the center of } C_b(\lambda),$$

then z'' must be the symmetric point of z' with respect to the real axis of the z-plane, namely $z'' = \overline{z}'$. On the other hand,

$$l_b\left(\lambda, -\frac{\varphi^{[\nabla]}(b,\lambda)}{\varphi(b,\lambda)}\right) = \infty.$$

Therefore, the center of the circle $C_b(\lambda)$ is given by

$$l_b\left(\lambda, -\frac{\overline{\varphi}^{[\nabla]}(b,\lambda)}{\overline{\varphi}(b,\lambda)}\right) = -\frac{W_b(\theta,\overline{\varphi})}{W_b(\varphi,\overline{\varphi})}.$$

The radius $r_b(\lambda)$ of the circle $C_b(\lambda)$ is equal to the distance between the center of $C_b(\lambda)$ and the point $l_b(\lambda,0)$ on the circle $C_b(\lambda)$. Hence

$$r_b(\lambda) = \left|\frac{\theta^{[\nabla]}(b,\lambda)}{\varphi^{[\nabla]}(b,\lambda)} - \frac{W_b(\theta,\overline{\varphi})}{W_b(\varphi,\overline{\varphi})}\right| = \left|\frac{W_b(\theta,\varphi)}{W_b(\varphi,\overline{\varphi})}\right|.$$

On the other hand, by virtue of (9), (10), $W_b(\theta,\varphi) = W_{\sigma(a)}(\theta,\varphi) = -1$. Further, by virtue of (8) and (9), we have

$$\begin{aligned}2v\int_a^b |\varphi(t,\lambda)|^2\,\Delta t &= iW_{\sigma(a)}(\varphi,\overline{\varphi}) - iW_b(\varphi,\overline{\varphi}) \\ &= -iW_b(\varphi,\overline{\varphi}), \qquad (15)\end{aligned}$$

where $v = Im\lambda$. Therefore, we obtain

$$r_b(\lambda) = \frac{1}{2|v| \int_a^b |\varphi(t,\lambda)|^2 \Delta t}, \quad Im\lambda = v \neq 0. \tag{16}$$

Since $\theta(b,\lambda)\varphi^{[\nabla]}(b,\lambda) - \varphi(b,\lambda)\theta^{[\nabla]}(b,\lambda) = W_b(\theta,\varphi) = -1 \neq 0$, the transformation (13) has a unique inverse which is given by

$$z = -\frac{\varphi^{[\nabla]}(b,\lambda)l + \theta^{[\nabla]}(b,\lambda)}{\varphi(b,\lambda)l + \theta(b,\lambda)}. \tag{17}$$

We shall now prove the following statement.

Lemma 3. If $v = Im\lambda > 0$, then the interior of the circle $C_b(\lambda)$ is mapped onto the lower half plane of the z-plane by the transformation (17), and, the exterior of the circle $C_b(\lambda)$ is mapped onto the upper half plane of the z-plane.

Proof. Since the real axis of the z-plane is the image of the circle $C_b(\lambda)$ by the transformation (17), the interior of $C_b(\lambda)$ is mapped onto either the upper half plane or the lower half plane of the z-plane, and further, the point at infinity of the l-plane is mapped onto the point $-\varphi^{[\nabla]}(b,\lambda)/\varphi(b,\lambda)$ of the z-plane. On the other hand, by making use of (15),

$$Im\left(-\frac{\varphi^{[\nabla]}(b,\lambda)}{\varphi(b,\lambda)}\right) = \frac{i}{2}\left\{\frac{\varphi^{[\nabla]}(b,\lambda)}{\varphi(b,\lambda)} - \frac{\overline{\varphi}^{[\nabla]}(b,\lambda)}{\overline{\varphi}(b,\lambda)}\right\}$$

$$= -\frac{i}{2}\frac{W_b(\varphi,\overline{\varphi})}{|\varphi(b,\lambda)|^2} = \frac{v}{|\varphi(b,\lambda)|^2}\int_{\sigma(a)}^b |\varphi(t,\lambda)|^2 \Delta t > 0.$$

This means that $-\varphi^{[\nabla]}(b,\lambda)/\varphi(b,\lambda)$ belongs to the upper half plane of the z-plane. Hence the point at infinity which is not contained in the interor of $C_b(\lambda)$ is mapped into the upper half plane. This proves the lemma.

Lemma 4. If $v = Im\lambda > 0$, then l belongs to the interior of the circle $C_b(\lambda)$ if and only if

$$\int_{\sigma(a)}^b |\theta(t,\lambda) + l\varphi(t,\lambda)|^2 \Delta t < -\frac{Iml}{v},$$

and, l lies on the circle $C_b(\lambda)$ if and only if

$$\int_{\sigma(a)}^b |\theta(t,\lambda) + l\varphi(t,\lambda)|^2 \Delta t = -\frac{Iml}{v}.$$

Proof. In view of Lemma 3, if $Im\lambda = v > 0$, then l belongs to the interior of the circle $C_b(\lambda)$ if and only if $Imz < 0$, that is, $i(z - \overline{z}) > 0$. From (17) it follows that

$$i(z-\overline{z}) = -i\frac{\varphi^{[\nabla]}(b,\lambda)l + \theta^{[\nabla]}(b,\lambda)}{\varphi(b,\lambda)l + \theta(b,\lambda)}$$
$$+i\frac{\overline{\varphi}^{[\nabla]}(b,\lambda)\overline{l} + \overline{\theta}^{[\nabla]}(b,\lambda)}{\overline{\varphi}(b,\lambda)\overline{l} + \overline{\theta}(b,\lambda)}$$
$$= \frac{iW_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi})}{|\varphi(b,\lambda)l + \theta(b,\lambda)|^2}.$$

Therefore, $Imz < 0$ if and only if

$$iW_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi}) > 0.$$

By formula (8) with $x = \theta + l\varphi$, we have

$$2v\int_{\sigma(a)}^b |\theta + l\varphi|^2 \Delta t$$
$$= i\{W_{\sigma(a)}(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi}) - W_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi})\}.$$

Further, by (9), (10) we have $W_{\sigma(a)}(\theta,\overline{\varphi}) = -1$, $W_{\sigma(a)}(\varphi,\overline{\theta}) = 1$, $W_{\sigma(a)}(\theta,\overline{\theta}) = W_{\sigma(a)}(\varphi,\overline{\varphi}) = 0$. Therefore

$$W_{\sigma(a)}(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi})$$
$$= W_{\sigma(a)}(\theta,\overline{\theta}) + \overline{l}W_{\sigma(a)}(\theta,\overline{\varphi})$$
$$+ lW_{\sigma(a)}(\varphi,\overline{\theta}) + |l|^2 W_{\sigma(a)}(\varphi,\overline{\varphi})$$
$$= l - \overline{l} = 2iIml.$$

Consequently

$$2v\int_{\sigma(a)}^b |\theta + l\varphi|^2 \Delta t = -2Iml - iW_b(\theta + l\varphi, \overline{\theta} + \overline{l}\overline{\varphi})$$

and the statements of the lemma follow.

Remark 5. It is easy to see that Lemma 4 also holds when $v = Im\lambda < 0$. In the both cases $v > 0$ and $v > 0$ the sign of Iml is opposite of the sign of v.

Lemma 6. If $v = Im\lambda \neq 0$, and $0 < b < b'$, then

$$\widetilde{C}_{b'}(\lambda) \subset \widetilde{C}_b(\lambda),$$

where $\widetilde{C}_b(\lambda)$ is the set composed of the circle $C_b(\lambda)$ and its interior.

Proof. If l belongs to the interior of the circle $C_{b'}(\lambda)$ or is on $C_{b'}(\lambda)$, then taking into account Lemma 4, we have

$$\int_{\sigma(a)}^b |\theta + l\varphi|^2 \Delta t \leq \int_{\sigma(a)}^{b'} |\theta + l\varphi|^2 \Delta t \leq -\frac{Iml}{v}.$$

Hence the lemma follows by using again Lemma 4.

Lemma 6 implies that, if $v = Im\lambda \neq 0$, then the set

$$\cap_{b>\sigma(a)}\widetilde{C}_b(\lambda) = C_\infty(\lambda)$$

is either a point or a closed circle with a nonzero finite radius.

Definition 7. According as $C_\infty(\lambda)$ is a point or a circle, the equation (3) is said to be in the limit point case or the limit circle case.

According to this definition, the classification seems to depend on the $p(t)$, $q(t)$, and λ. However, it is independent of λ and depends only on $p(t)$, $q(t)$, as is shown in the next section.

Let $m = m(\lambda)$ be the limit point $C_\infty(\lambda)$ or any point on the limit circle $C_\infty(\lambda)$. Then for any $b \in (a,\infty)_{\mathbb{T}}$, we have

$$\int_{\sigma(a)}^b |\theta(t,\lambda) + m(\lambda)\varphi(t,\lambda)|^2 \Delta t \leq -\frac{Imm(\lambda)}{v}.$$

Hence

$$\int_{\sigma(a)}^\infty |\theta(t,\lambda) + m(\lambda)\varphi(t,\lambda)|^2 \Delta t \leq -\frac{Imm(\lambda)}{v}.$$

Denote by $L^2_\Delta(a,\infty)$ the space of all complex-valued Δ-measurable (see [6]) functions f on $[a,\infty)_{\mathbb{T}}$ such that

$$\int_a^\infty |f(t)|^2 \Delta t < \infty.$$

Similarly, we can introduce the space $L^2_\nabla(a,\infty)$.

Thus, we have obtained the following theorem.

Theorem 8. For all nonreal values of λ there exists a solution

$$\psi(t,\lambda) = \theta(t,\lambda) + m(\lambda)\varphi(t,\lambda)$$

of Eq. (3) such that $\psi \in L^2_\Delta(a,\infty)$.

In the limit circle case the radius $r_b(\lambda)$ tends to a finite nonzero limit as $b \to \infty$. Then (16) implies that in this case also $\varphi \in L^2_\Delta(a,\infty)$. Therefore, in the limit circle case all solutions of Eq. (3) belong to $L^2_\Delta(a,\infty)$ for $Im\lambda \neq 0$ because in this case both $\varphi(t,\lambda)$ and $\theta(t,\lambda) + m(\lambda)\varphi(t,\lambda)$ belong to $L^2_\Delta(a,\infty)$, and this identifies the limit circle case. We will see below in Theorem 9 that in the limit circle case all solutions of Eq. (3) belong to $L^2_\Delta(a,\infty)$ also for all real values of λ. In the limit point case, $r_b(\lambda)$ tends to zero as $b \to \infty$, and from (16) this implies that $\varphi(t,\lambda)$ is not of class $L^2_\Delta(a,\infty)$. Therefore in this situation there is only one soluton of class $L^2_\Delta(a,\infty)$ for $Im\lambda \neq 0$. Note that in the limit point case the equation may not have any nontrivial solution of class $L^2_\Delta(a,\infty)$ for real values of λ. For example, (for $\lambda = 0$), the equation $-y^{\nabla\Delta} = 0$ has the general solution $y(t) = c_1 + c_2 t$ and evidently this solution belongs to $L^2_\Delta(a,\infty)$ only for $c_1 = c_2 = 0$.

3. INVARIANCE OF THE LIMIT POINT AND LIMIT CIRCLE PROPERTIES

In the previous section the expressions "limit point case" and "limit circle case" were applied to particular values of λ; but in fact if the limit is a circle for any complex λ, it is a circle for every complex λ. In the present section we prove this property.

Theorem 9. If every solution of $Ly = \lambda_0 y$ is of class $L^2_\Delta(a,\infty)$ for some complex number λ_0, then for arbitrary complex number λ every solution of $Ly = \lambda y$ is of class $L^2_\Delta(a,\infty)$.

Proof. It is given that two linearly independent solutions $y_1(t)$ and $y_2(t)$ of $Ly = \lambda_0 y$ are of class $L^2_\Delta(a,\infty)$. Let $\chi(t)$ be any solution of $Ly = \lambda y$, which can be written as

$$Ly = \lambda_0 y + (\lambda - \lambda_0)y.$$

By multiplying y_1 by a constant if necessary (to achieve $W_t(y_1,y_2) = 1$) a variation of constants formula (6) yields

$$\chi(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$+(\lambda-\lambda_0)\int_c^t [y_1(t)y_2(s) - y_1(s)y_2(t)]\chi(s)\Delta s, \tag{18}$$

where c_1, c_2 are constants and c is any fixed point in $[a,\infty)_{\mathbb{T}}$. Let us introduce the notation

$$\|\chi\|_{c,t} = \left\{\int_c^t |\chi(s)|^2 \Delta s\right\}^{1/2}$$

for $t \in [a,\infty)_{\mathbb{T}}$ with $t \geq c$. Next, let M be such that $\|y_1\|_{c,t} \leq M$, $\|y_2\|_{c,t} \leq M$ for all $t \in [a,\infty)_{\mathbb{T}}$ with $t \geq c$; such a constant M exists because y_1 and y_2 are of class $L^2_\Delta(a,\infty)$. Then the Cauchy-Schwarz inequality gives

$$\left|\int_c^t [y_1(t)y_2(s) - y_1(s)y_2(t)]\chi(s)\Delta s\right|$$
$$\leq |y_1(t)| \int_c^t |y_2(s)|\,|\chi(s)|\,\Delta s$$
$$+ |y_2(t)| \int_c^t |y_1(s)|\,|\chi(s)|\,\Delta s$$
$$\leq M(|y_1(t)| + |y_2(t)|)\,\|\chi\|_{c,t}\,.$$

Using this in (18) yields

$$|\chi(t)| \leq |c_1|\,|y_1(t)| + |c_2|\,|y_2(t)|$$
$$+ |\lambda-\lambda_0|\, M(|y_1(t)| + |y_2(t)|)\,\|\chi\|_{c,t}\,.$$

Hence applying the Minkowski inequality (note that the usual Cauchy-Schwarz and Minkowski inequalities hold on time scales, see [3]), we get

$$\|\chi\|_{c,t} \leq (|c_1| + |c_2|)M + 2\,|\lambda-\lambda_0|\,M^2\,\|\chi\|_{c,t}\,.$$

If c is chosen large enough so that

$$|\lambda-\lambda_0|\,M^2 < \frac{1}{4},$$

then

$$\|\chi\|_{c,t} \leq 2(|c_1| + |c_2|)M.$$

Since the right side of this inequality is independent of t, it follows that $\chi \in L^2_\Delta(a,\infty)$ and the theorem is proved.

4. A CRITERION FOR THE LIMIT POINT CASE

In this section we present a simple criterion for the limit point case. For the usual Sturm-Liouville equation it was established earlier by Putnam [10].

Theorem 10. If p is arbitrary and $q \in L^2_\Delta(a,\infty)$, then Eq. (3) is in the limit point case.

Proof. It is sufficient to show that the equation

$$-[p(t)y^\nabla(t)]^\Delta + q(t)y(t) = 0, \quad t \in (a,\infty)_{\mathbb{T}}, \tag{19}$$

does not have two linearly independent solutions belonging to $L^2_\Delta(a,\infty)$.

If y is such a solution, then, because of the condition $q \in L^2_\Delta(a,\infty)$, the function $(y^{[\nabla]})^\Delta = qy$ belongs to $L^1_\Delta(a,\infty)$ by the Cauchy-Schwarz inequality. Therefore the limit

$$\lim_{t\to\infty} y^{[\nabla]}(t) = y^{[\nabla]}(t_0) + \int_{t_0}^\infty (y^{[\nabla]})^\Delta(s)\Delta s$$

exists and is finite. Hence the function $y^{[\nabla]}(t)$ is bounded as $t \to \infty$.

Now let y_1, y_2 be two linearly independent solutions of Eq. (19); then

$$y_1(t)y_2^{[\nabla]}(t) - y_1^{[\nabla]}(t)y_2(t) = c \neq 0.$$

If $y_1 \in L^2_\Delta(a,\infty)$ and $y_2 \in L^2_\Delta(a,\infty)$, then $y_1^{[\nabla]}$ and $y_2^{[\nabla]}$ are bounded, and so the function $y_1 y_2^{[\nabla]} - y_1^{[\nabla]} y_2 = c \neq 0$ also belongs to $L^2_\Delta(a,\infty)$, which is impossible. The theorem is proved.

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