On vector equilibrium Problems

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Abstract: In this paper, we consider strong form of a vector equilibrium problem and establish some existence results for solutions of such problem in the setting of topological vector spaces. We provide several coercivity conditions under which strong vector equilibrium problem has a solution. Our results generalize and extend the results of Bianchi and Pini [Coercivity conditions for equilibrium problems, J. Optim. Theory Appl., 124 (2005) 79-92] to the topological vector space setting.

Keywords: Vector equilibrium problems, existence results, coercivity conditions, generalized monotonicity, KKM-Fan lemma.

1. INTRODUCTION

Let X and Y be two Hasudorff topological vector spaces, K a nonempty convex subset of X, C a closed convex pointed cone in Y and $f : K \times K \to \Re$ be a real-valued bifunction. The equilibrium problem (for short, EP) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in K. \tag{1}$$

It is well known that EP is a unified model of several problems, namely, variational inequality problem, complementarity problem, fixed point problem, saddle point problem, Nash equilibrium problem, etc.. For further detail, we refer to (3) and references therein.

Motivated by the extensions of variational inequalities to vector variational inequalities, many authors generalized EP to the vector case in the following ways:

Find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -\text{int}C, \quad \forall y \in K, \tag{2}$$

Find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \in C, \quad \forall y \in K, \tag{3}$$

where int$C$ denotes the interior of C and in this case, we assume that int$C \neq \emptyset$, and $f : K \times K \to Y$.

These problems are called vector equilibrium problems (for short, VEP)

The following problem, termed as dual vector equilibrium problem (for short, DVEP) is closely related to VEP (3):

Find $\bar{x} \in K$ such that

$$f(y, \bar{x}) \in -C, \quad \forall y \in K.$$

We denote by S_K and S_K^D the set of all solutions of VEP (3) and DVEP, respectively. Recently, Aussel and Hadisavvas (2) introduced the concept of local solutions of variational inequalities. Bianchi and Pini (4) extended this concept of local solutions to EP in the setting of reflexive Banach spaces. We further extend this concept for vector equilibrium problems in the setting of topological vector spaces.

The sets of local solutions for VEP (3) and DVEP are denoted by $S_{K,loc}$ and $S_{K,loc}^D$, respectively. We define $S_{K,loc}$ and $S_{K,loc}^D$ as follows:

$S_{K,loc}^D = \{x \in K : \exists V$ open neighborhood of x such that $f(y, x) \in -C, \forall y \in V \cap K\}$,
$S_{K,loc} = \{x \in K : \exists V$ open neighborhood of x such that $f(x, y) \in C, \forall y \in V \cap K\}$.

$Obviously, S_K^D \subseteq S_{K,loc}^D and S_K \subseteq S_{K,loc}.$

In most the papers appeared in the literature on the different aspects, VEP (2) is considered and studied. It is worth to mention that the weak (strong, respectively) Pareto optimal solution of a vector optimization problem can be studied by using VEP (2) ((3), respectively). There are few papers appeared on the existence of solutions of VEP (3), see for example (2) and references therein.

The main motivation of this paper is to introduce several coercivity conditions and to study the existence of solutions of VEP (3) under these coercivity conditions. Our results generalize and extend the results of Bianchi and Pini (4) to the topological vector space setti

2. MAIN RESULTS

The following concept of upper sign continuity is vector version of the the upper sign continuity was introduced by Bianchi and Pini (4) which extends the earlier notion in the framework of variational inequalities in (5).

Definition 1. Let $y \in K$ be any fixed element. A function $x \to f(x, y)$, $x \in K$, is said to be *upper sign continuous* if for every $x \in K$,

$$f(u,y) \in C, \quad \forall u \in \ ]x,y[\quad \Rightarrow f(x,y) \in C,$$

where $]x,y[$ denotes the line segment joining x and y except the end points x and y.

We can analogues define hemicontinuity of f so that it implies upper sign continuity of f and $-f$.

Definition 2. A bifunction $f : K \times K \to Y$ is said to be

(i) *C-pseudomonotone* if $\forall x, y \in K$,

$$f(x,y) \in Y \setminus (-C) \quad \Rightarrow \quad f(y,x) \in -C \setminus \{0\};$$

(ii) *C-quasimonotone* if $\forall x, y \in K$,

$$f(x,y) \in Y \setminus (-C) \quad \Rightarrow \quad f(y,x) \in -C;$$

(iii) *C-properly quasimonotone* if, for all $\{x_1, \ldots, x_n\} \subseteq K$ and for all $x \in \text{co}\{x_1, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that $f(x_i, x) \in -C$, where $\text{co}\{x_1, \ldots, x_n\}$ denotes the convex hull generated by $x_1, \ldots, x_n$.

It is clear from the definition that C-pseudomonotonicity of f implies C-quasimonotonicity. But in general there is no relationship between C-properly quasimonotonicity and quasimonotonicity or pseudomonotonicity.

The following proposition is a vector version of a proposition given in (3) which provides a criteria for the C-proper quasimonotonicity of a function.

Proposition 3. Let $f : K \times K \to Y$ be a vector-valued bifunction such that one of the following conditions holds:

(i) The set $\{x \in K : f(x,y) \in Y \backslash (-C \backslash \{0\})\}$ is convex and $f(y,y) \in -C$ for all $y \in K$;
(ii) The set $\{y : f(x,y) \in -C \backslash \{0\}\}$ is convex and f is C-pseudomonotone such that $f(x,x) = 0$ for all $x \in K$.

Then, f is C-properly quasimonotone.

Proof. Suppose that (i) holds and assume contrary that there exist $x_1, \ldots, x_n \in K$ and $\tilde{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \tilde{x}) \in Y \backslash (-C)$ for $i = 1, 2, \ldots, n$, Therefore, by (i) we have $f(\tilde{x}, \tilde{x}) \in Y \backslash (-C)$ and so $f(\tilde{x}, \tilde{x}) \notin -C$ which is a contradiction of our assumption that $f(y,y) \in -C$ for all $y \in K$.

Suppose that (ii) holds but there exist $x_1, \ldots, x_n \in K$ and $\tilde{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \tilde{x}) \in Y \backslash (-C)$ for $i = 1, 2, \ldots, n$. Since f is C-pseudomonotone, we have $f(\tilde{x}, x_i) \in -C \backslash \{0\}$ and so $f(\tilde{x}, \tilde{x}) \in -C \backslash \{0\}$ since the set $\{y : f(x,y) \in -C \backslash \{0\}\}$ is convex. Therefore, $f(\tilde{x}, \tilde{x}) \neq 0$ which is a contradiction of our assumption that $f(x,x) = 0$ for all $x \in K$.

The following lemma plays a key role in this section.

Lemma 4. Let K be a nonempty convex subset of X and $f : K \times K \to Y$ be a vector-valued bifunction such that the following conditions hold:

(i) $f(x,x) \in C$ for all $x \in K$;
(ii) For all $x \in K$, the mapping $x \mapsto f(x,y)$ is upper sign continuous;
(iii) If $f(x,y) \in Y \backslash C$ and $f(x,z) \in -C$, then $f(x,u) \in Y \backslash C$, for all $u \in \ ]y,z[$.

Then, $S^D_{K,loc} \subseteq S_K$.

Proof. Let $z \in S^D_{K,loc}$. In order to show that $z \in S_K$, we assume contrary that there exists $y \in K$ such that $f(z,y) \in Y \backslash C$. From the definition of $S^D_{K,loc}$, there exists an open neighborhood V of z such that $f(v,z) \in -C$ for all $v \in K \cap V$. Since $V - z$ is a neighborhood of 0, there exists $t_0 \in]0,1[$ such that $t(y-z) \in V - z$ for all $0 < t \leq t_0$. Let $\bar{y} = z + t_0(y-z)$ and $y_t = (1-t)z + t\bar{y} \in [z, \bar{y}]$ for $t \in [0,1]$. Then $y_t \in K \cap V$, since $y_t = (1-t)z + tz + t\, t_0(y-z) = z + t\, t_0(y-z)$ and $t\, t_0(y-z) \in V - z$. Hence, by our assumption on z, $f(y_t, z) \in -C$ and by condition (i), $f(z,z) = 0$. Now we will show that $f(u, \bar{y}) \in C$ for all $u \in \ ]z, \bar{y}[$. Indeed, if $f(u, \bar{y}) \in Y \backslash C$ for some $u \in \ ]z, \bar{y}[$, then as $f(u,z) \in -C$, we deduce from (iii) that $f(u,v) \in Y \backslash C$ for all $v \in]z, \bar{y}[$ and in particular $f(u, v = u) = 0 \in Y \backslash C$. Hence $0 \notin C$ which contradicts the fact that $0 \in C$ since C is a pointed cone. Therefore, $f(u, \bar{y}) \in C$ for all $u \in \ ]z, \bar{y}[$. Thus by (ii), we have

$$f(z, \bar{y}) \in C. \tag{4}$$

Since $f(z,z) = 0$ and $f(z,y) \in Y \backslash C$, from (iii) it follows that $f(z, \bar{y}) \in Y \backslash C$ which contradicts (4).

Example. Let $X = Y = \Re$, $K = [-1, 1]$, $C = [0, \infty)$ and $f : [-1,1] \times [-1,1] \to \Re$ be defined as

$$f(x,y) = \begin{cases} 0 & if \quad (x,y) \in \{0\} \times \left[\dfrac{-1}{2}, \dfrac{1}{2}\right] \quad or \quad x = y, \\ -1 & otherwise. \end{cases}$$

It is clear that $f(x,x) = 0$, for all $x \in K$ and if $f(u,y) \geq 0$, for all $u \in]x,y[$, then $u = 0$, for all $u \in]x,y[$, which is impossible. This shows that the mapping $x \to f(x,y)$ is upper sign continuous mapping for each fixed $y \in K$. From the relations $f(\frac{1}{4}, \frac{1}{3}) < 0, f(\frac{3}{4}, \frac{1}{3}) < 0$, one can verify that the statement $f(u, \frac{1}{3}) < 0$, does not true for all $u \in]\frac{1}{4}, \frac{3}{4}[$, for example take $u = \frac{1}{3} \in]\frac{1}{4}, \frac{3}{4}[$, and so the example does not fulfill the condition (iii) of Lemma 4 Moreover, the result in Lemma 4 is not true for this example, since $x_0 = 0 \in S_{[\frac{-1}{2}, \frac{1}{2}]}$ and $S_K = \emptyset$.

Remark 5. (a) If the mapping $y \mapsto f(x,y)$ is C-convex, that is,

$$tf(x,y) + (1-t)f(x,z) - f(x, ty + (1-t)z) \in C, \quad \forall y, z \in K, \forall t \in [0,1]. \tag{5}$$

then condition (ii) of Lemma 4 holds. To see this, let $f(x,y) \in Y \backslash C$ and $f(x,z) \in -C$. Since $Y \backslash C$ and $-C$ are cone, we have $tf(x,u) \in Y \backslash C$ and $(1-t)f(x,z) \in -C$. Since $(Y \backslash C) - C \subseteq Y \backslash C$, we have

$$tf(x,u) + (1-t)f(x,z) \in Y \backslash C. \tag{6}$$

From (5), (6) and the fact $(Y \backslash C) - C \subseteq Y \backslash C$, we have

$$f(x, ty + (1-t)z) \in Y \backslash C, \quad \forall t \in [0,1].$$

This shows that condition (ii) of Lemma 4 holds.

(b) Lemma 4 improves and extends Lemma 2.4 in (8) and Lemma 2.1 in (4) to vector-valued bifunctions.

(c) If for all $x, y \in K$, $f(x,y) \in Y\backslash(-C\backslash\{0\})$ implies $f(y,x) \in -C$, then $S_K \subseteq S_K^D$. Therefore, under this assumption, we have $S_K^D = S_{K,loc}^D = S_K$. Thus, if $Y = \Re$ and $C = [0, \infty)$, we deduce Proposition 2.5 in (8).

Throughout the paper, for a nonempty set A, we denote by 2^A (respectively, $\mathcal{F}(A)$) the family of all (respectively, nonempty finite) subsets of A. If A is a nonempty subset of a topological space, $\overline{A}$ and *int* A denote the closure and interior of A, respectively.

Let K be a convex subset of a vector space X. Then a set function $F : K \to X$ is called a KKM mapping if for each nonempty finite subset A of K, co$A \subseteq F(A)$, where co$A$ denotes the convex hull of A and $F(A) = \bigcup\{F(x) : x \in A\}$.

The following lemma will be used in the sequel which is a special case of Fan-KKM principle (6).

Lemma 6. Let X be a nonempty subset of a topological vector space E and $F : X \to E$ be a KKM mapping with closed values. Assume that there exist a nonempty compact convex subset B of X such that $\bigcap_{x\in B} F(x)$ is compact. Then $\overline{\bigcap_{x\in X} F(x)} \neq \emptyset$.

We derive the existence result for a solution of DVEP.

Theorem 7. Assume that the following conditions hold:

(i) For each $A \in \mathcal{F}(K)$ and $x \in coA\backslash A$, there exists $y \in K$ such that $f(y,x) \in -C$;
(ii) For all $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is closed in K;
(iii) There exist a nonempty compact subset D and a nonempty compact convex subset B of K such that for each $y \in K\backslash D$, there exists $x \in B$ such that $f(x,y) \in Y\backslash -C$.

Then, the solution set S_K^D of DVEP is nonempty and compact.

Proof. For all $x \in K$, we define set function $\Gamma : K \to K$ as

$$\Gamma(x) = \{y \in K : f(x,y) \in -C\}.$$

It follows from condition (i) that Γ is a KKM mapping. Condition (iii) implies that $\bigcap_{x\in K}\Gamma(x) \subseteq \bigcap_{x\in B}\Gamma(x) \subseteq D\bigcap K$ which is closed and compact in K. Therefore by Lemma 5, $S_K^D = \bigcap_{x\in K}\Gamma(x) \neq \emptyset$.

Proposition 8. If for each $x \in K$, the set $\{y \in K : f(x,y) \in -C\}$ is closed in K and convex, then either f is C-properly quasimonotone or $S_{K,loc}^D \neq \emptyset$.

Proof. Assume that there exist $x_1, \ldots, x_n \in K$ and $\bar{x} \in \text{co}\{x_1, x_2, \ldots, x_n\}$ such that $f(x_i, \bar{x}) \in Y\backslash(-C)$, for all $i = 1, 2, \ldots, n$. Then, there exists an open neighborhood V of $\bar{x}$ such that

$$f(x_i, y) \in Y\backslash(-C), \quad \forall y \in K \cap V, \ i = 1, 2, \ldots, n.$$

Hence from the $C-$ quasimonotonicity of f we have $f(y, x_i) \in Y - C$ for all $i = 1, 2, \ldots, n$ and $y \in K\cap V$ and so by our assumption, that is the set $\{y \in K : f(x,y) \in -C\}$ is convex, we get

$$f(y, \bar{x}) \in -C, \quad \forall y \in K \cap V.$$

Therefore, $\bar{x} \in S_{K,loc}^D$.

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