Approximate Polynomial Solution of a Nonlinear Differential Equation Using a Taylor-type Method

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Abstract: The paper proposes an approximate analytical polynomial solution for a nonlinear problem of the type : $x''=F(x',x,t)$ on the [a,b] interval with initial conditions of the type : $x'(a)=x_1,\ x(a)=x_0$, where F is a continuously differentiable real function. The approximate analytical solution is expressed in terms of Taylor polynomials, whose coefficients are determined by solving a nonlinear system associated to the problem. The performance of the method is illustrated by two numerical examples.

Keywords**:** Nonlinear differential equation; Analytical approximate polynomial solution; Taylor polynomial; Duffing equation; Van der Pol equation.

1. INTRODUCTION

For many differential equations which model physical phenomena it is usually difficult, or even impossible to find an exact solution. In such a situation, an approximate solution must be found.

An often used category of approximate solution consists of polynomial solutions, which presents certain advantages over other types of approximate solutions: compact expression, facile manipulation in subsequent computations etc.

In order to obtain approximate polynomial solutions, various methods are used, such as : methods based on Taylor polynomials (Funaro 1992, Gulsu and Sezer 2006), Bernstein polynomials (Bhatti and Bracken 2007) and Chebyshev polynomials (Chena *et.al.* 2005), variational methods (Zhan-Hua 2008), homotopy methods (Belandez *et.al.* 2007, He 2008), Adomian decomposition methods (Al-Hayani and Casasus 2006, Shawagfeh and Kaya 2004), collocation-type methods (Von Stryk 1993, D'Ambrosio *et.al.* 2009), differential transform methods (Hassan and Ertürk 2007, Abazari and Abazari 2009) and harmonic analysis methods (Bota 2008).

In the present paper we propose an approximate polynomial solution for a differential equation of the type:

$$x''=F(x',x,t) \tag{1}$$

with condition of the type: $x'(a)=x_1,\ x(a)=x_0$, where F is a continuously differentiable real function defined on the $[a,b]$ interval and x_0, x_1, a, b are real constants.

Equations of this type have multiple applications in various fields such as nonlinear mechanics, electrodynamics, nonlinear dynamical systems, astrophysics, quantum mechanics etc.

Our approximate polynomial solution for the equation (1) has its roots in the regular differential transform method (Ertürk 2007a b and c, Ebaid and Ali 2009, Ahmad 2009, Odibat *et.al.* 2010, Iscan and Ongun 2009), and the coefficients are determined using an original method, presented in the following.

2. METHOD OF SOLUTION

We consider the solution of the equation (1) expressed as:

$$\tilde{x}(t)=\sum_{k=0}^{n} c_k \cdot t^k , \tag{2}$$

The parameters $c_0, c_1, ..., c_n$ are calculated using the steps outlined in the following.

By substituting the approximate solution $\tilde{x}(t)=\sum_{k=0}^{n} c_k \cdot t^k$ in the equation (1) we obtain the following expression:

$$R(\tilde{u},t) \doteq \tilde{x}''(t)-F(\tilde{x}',\tilde{x},t) \tag{4}$$

If we could find the constants $c_0, c_1, ..., c_n$ such that $R(\tilde{x},t)=0$ for any t in the $[a,b]$ interval, then by substituting $c_0, c_1, ..., c_n$ in (2) we obtain the exact solution of (1). In general this situation is rarely encountered. The best approximate solution of the problem (1,2) is thus the one which minimizes the value of $|R(t,\tilde{x})|$.

We remark that, due to the fact that *R* is a continuous function on the [a, *b*] interval, the following equivalence is easy to prove:

$$|R(t,\tilde{x})| = 0 \quad \text{if and only if} \quad \int_a^b R^2(t,\tilde{x})\,dt = 0 \cdot \qquad (4)$$

The initial conditions $x(a) = x_0$ and $x'(a) = x_1$ lead to the system consisting of the equations $\tilde{x}(a) = x_0$, $\tilde{x}'(a) = x_1$, thus allowing us to calculate the expressions of the coefficients c_0 and c_1 as functions of the other coefficients $c_2, ..., c_n$. In order to calculate these other *n* - 1 coefficients $c_2, ..., c_n$, we observe that by considering the equidistant partition of the [*a*, *b*] interval : $a = b_0 < b_1 < b_2 < ... b_n = b$, the equality $\int_a^b R^2(t,\tilde{x})\,dt = 0$ is equivalent with the following nonlinear system of equations in the unknowns $c_2, ..., c_n$:

$$\begin{cases} \int_a^{b_1} R^2(t,\tilde{x})\,dt = 0 \\ \int_{b_1}^{b_2} R^2(t,\tilde{x})\,dt = 0 \\ \vdots \\ \int_{b_{n-1}}^{b_n} R^2(t,\tilde{x})\,dt = 0 \end{cases} \qquad (5)$$

An approximate solution of this system, together with the expressions of the coefficients c_0 and c_1 as functions of $c_2, ..., c_n$ will give us, by means of the relation (2), an approximate solution of equation (1), $\tilde{x}(t) = \sum_{k=0}^{n} c_k t^k$.

If the approximate solution of the system (5) is computed using a numerical method which allows us to estimate the error ε , (such as, for example, a Newton-type method), then, if $c_2, ..., c_n$ is an approximate numerical solution of (5) and $c_2^0, ..., c_n^0$ is its corresponding exact solution, we have $c_i - \varepsilon \le c_i^0 \le c_i + \varepsilon, \quad i = 1, ..., n$.

On the other hand, the error-function $\mathrm{E}(t, c_2, ... c_n)$ obtained by replacing the exact solution $x(t)$ with the approximate solution $\tilde{x}(t) = \sum_{k=0}^{n} c_k t^k$ can be estimated as $\mathrm{E}(t, c_2, ... c_n) = |R(t,\tilde{x})|$. The maximal error E in this case can be computed as $\mathrm{E} = \max_{t\in[a,b],\, s_i\in[c_i-\varepsilon,\, c_i+\varepsilon],\ i=2,...,n} \mathrm{E}(t, s_2, ... s_n)$.

The method described above has the advantage, in comparison with other methods, that it can be applied not only for weakly-nonlinear equations but also for strong nonlinear equations.

For each of the examples considered in this paper we will give an estimate of the error, computed as

$$\tilde{R}(t) = \tilde{x}''(t) - F(\tilde{x}', \tilde{x}, t) . \qquad (6)$$

$\tilde{R}$ represents the error obtained by replacing the exact solution x with the approximate solution $\tilde{x}$.

3. EXAMPLES AND NUMERICAL RESULTS

3.1 Example 1

We consider the initial value problem corresponding to the Duffing differential equation (Sahmsul Alam *et. al.* 2006):

$$\begin{cases} \ddot{x} + 2k\dot{x} + w^2 x = -\varepsilon x^3 \\ x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \qquad (7)$$

on [0,1], where $k = \frac{1}{\sqrt{2}}, \ \varepsilon = w = 1$.

The Duffing equation, which describes the motion of a forced oscillator, is a nonlinear second-order equation of the type (1). It does not admit an exact analytical solution, and various approximate methods were employed in its study.

Using our method, the *cubic approximate analytical solution* is :

$$x_{app}(t) = 1 - 0.908452t^2 + 0.396015t^3.$$

The following plot contains the graphical representation of this polynomial (solid line) together with the corresponding numerical solution of equation (7) computed using the software Mathematica 6 (dashed line).

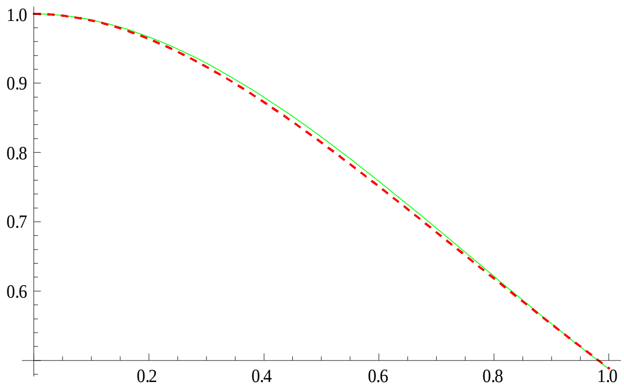

Fig. 1. Comparison between the numerical solution (dashed line) and the cubic polynomial approximation (solid line).

Next we present the graphical representation of the error R.

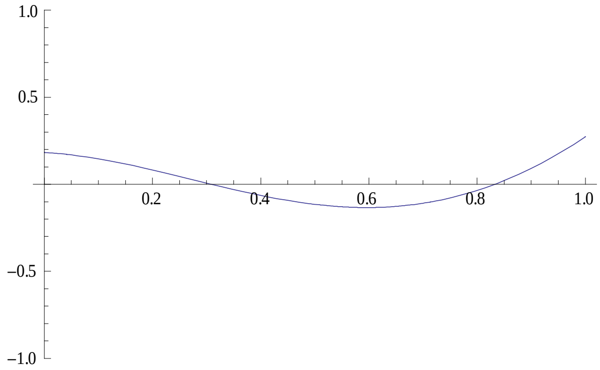

Fig. 2. Graphical representation of the error for the case of a cubic polynomial approximation.

We can observe that the maximal value of the error in this case is less than 0.2.

The *5th degree polynomial approximate analytical* solution :

$$x_{app}(t) = 1 - 1.01309t^2 + 0.557159t^3 - 0.0629464t^4 - 0.0492236t^5.$$

The following plot contains the graphical representation of this polynomial (solid line) together with the corresponding exact solution of equation (7) (dashed line).

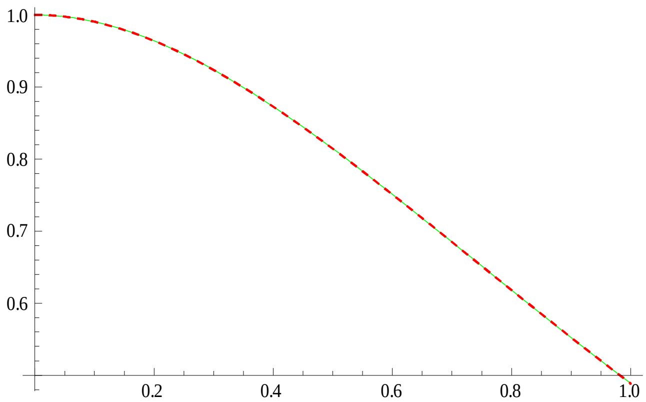

Fig. 3. Comparison between the exact solution (dashed line) and the 5th degree polynomial approximation (solid line).

It can be observed again that the graphical representations are practically overlapping.

The graphical representation of the error R is :

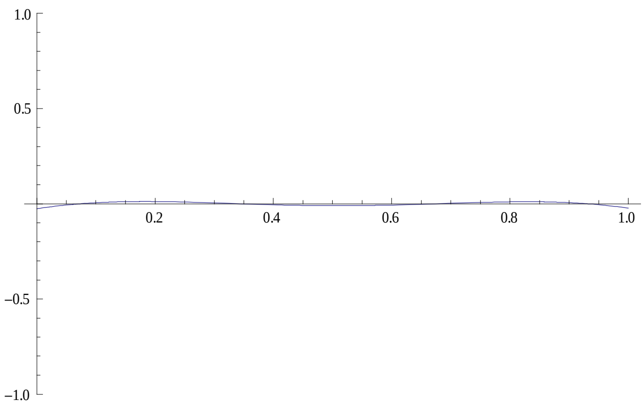

Fig. 4. Graphical representation of the error for the case of a 5th degree polynomial approximation.

We can observe that the maximal value of the error in this case is less than 0.03.

3.2 Example 2

We consider the initial value problem on the on [0,1] interval corresponding to the Van der Pol differential equation (Liao 2004) :

$$\begin{cases} \ddot{x} + x - \varepsilon(1 - x^2)\dot{x} = 0 \\ x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \tag{8}$$

The well-known Van der Pol equation models a damped oscillator and various approximate analytical solutions were proposed, but unfortunately most of these approximate solutions only work for small values of the parameter ε. In contrast, our method does not depend on ε.

3.2.1 The case $\varepsilon = 1$

First we apply our method for the value $\varepsilon = 1$.

The *cubic approximate analytical solution* is :

$$x_{app}(t) = 1 - 0.460506t^2 + 0.0394723t^3.$$

The following plot contains the graphical representation of this polynomial (solid line) together with the corresponding numerical solution of equation (8) computed using Mathematica 6 (dashed line).

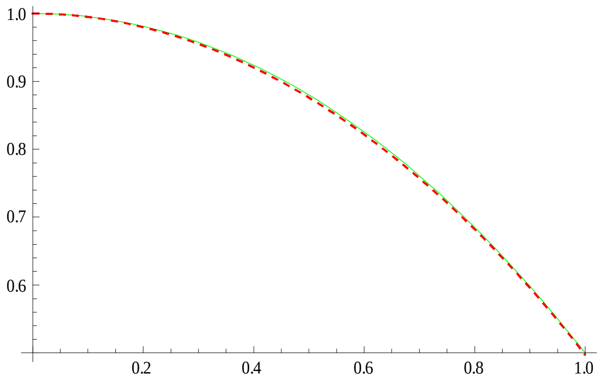

Fig. 5. Comparison between the numerical solution (dashed line) and the cubic polynomial approximation (solid line).

Next we present the graphical representation of the error R.

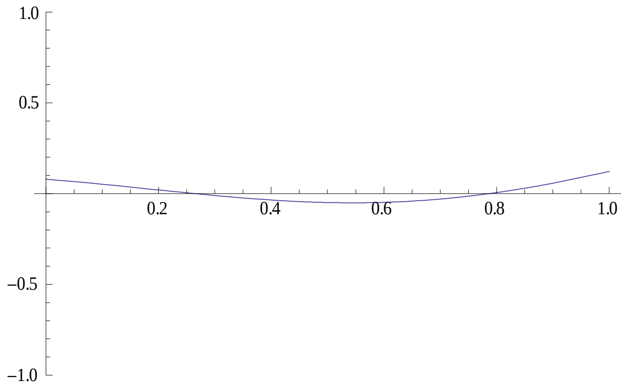

Fig. 6. Graphical representation of the error for the case of a cubic polynomial approximation.

We can observe that the maximal value of the error in this case is less than 0.01.

The *5th degree polynomial approximate analytical* solution is:

$$x_{app}(t) = 1 - 0.717481t^2 + 0.599461t^3 - 0.5108754t^4 + 0.12751t^5.$$

The following plot contains the graphical representation of this polynomial (solid line) together with the corresponding exact solution of equation (8) (dashed line).

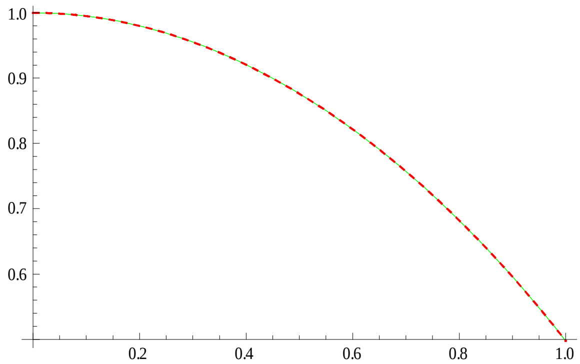

Fig. 7. Comparison between the exact solution (dashed line) and the 5th degree polynomial approximation (solid line).

It can be observed again that the graphical representations are practically overlapping.

The graphical representation of the error R is :

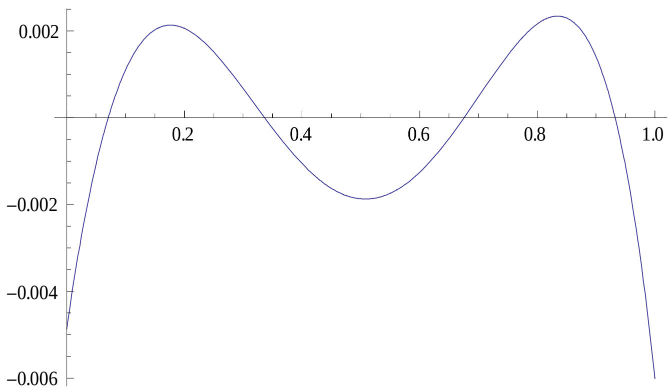

Fig. 8. Graphical representation of the error for the case of a 5th degree polynomial approximation.

We can observe that the maximal value of the error in this case is less than 0.005.

3.2.1 The case $\varepsilon = 2$

In order to show that our method is also valid for larger values of the parameter ε, next we apply our method for the value $\varepsilon = 2$.

The *5th degree polynomial approximate analytical* solution in this case is:

$$x_{app}(t) = 1 - 0.492807t^2 - 0.040983t^3 + 0.111263t^4 - 0.132975t^5.$$

The following plot contains the graphical representation of this polynomial (solid line) together with the corresponding numerical solution of equation (8) computed using Mathematica 6 (dashed line).

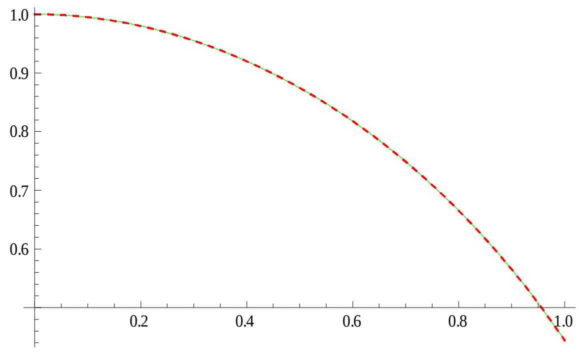

Fig. 9. Comparison between the numerical solution (dashed line) and the cubic polynomial approximation (solid line).

Next we present the graphical representation of the error R.

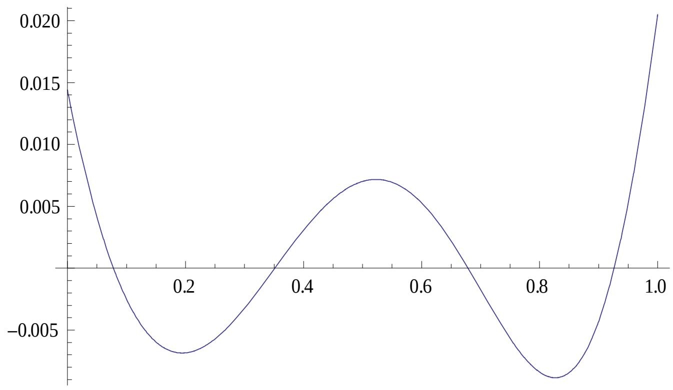

Fig. 10. Graphical representation of the error for the case of a cubic polynomial approximation.

We can observe that the maximal value of the error in this case is less than 0.02.

As it is to be expected, the error is larger than the corresponding one form the case $\varepsilon = 1$ (for a polynomial of the same degree).

Still, if a smaller error is desired, we only have to choose a higher degree of the polynomial. Thus, the *7th degree polynomial approximate analytical* solution is:

$$x_{app}(t) = 1 - 0.500147\,t^2 + 0.00132933\,t^3 + 0.0395706\,t^4 - \\ - 0.109087\,t^5 + 0.0347914\,t^6 - 0.0219653\,t^7.$$

The following plot contains the graphical representation of this polynomial (solid line) together with the corresponding exact solution of equation (8) (dashed line).

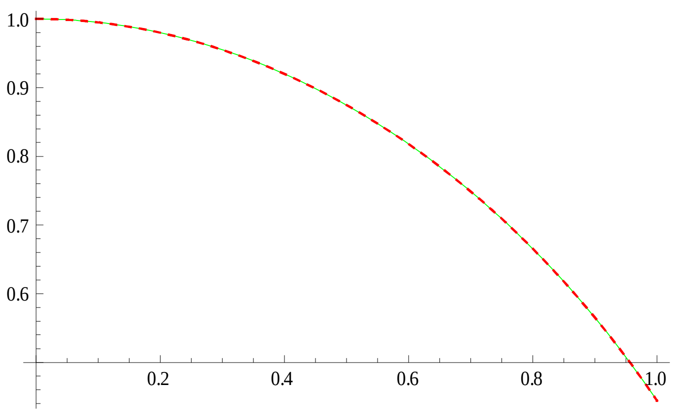

Fig. 11. Comparison between the exact solution (dashed line) and the 5th degree polynomial approximation (solid line).

It can be observed again that the graphical representations are practically overlapping.

The graphical representation of the error R is :

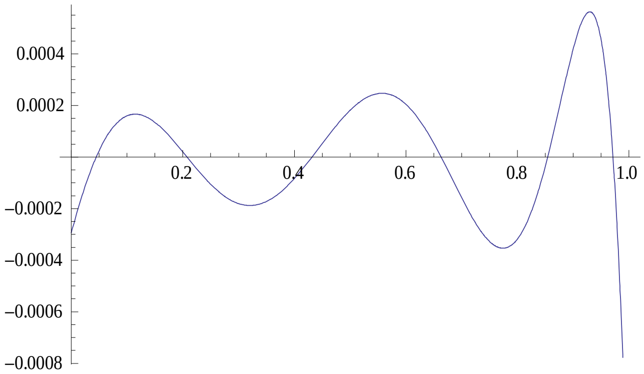

Fig. 12. Graphical representation of the error for the case of a 5th degree polynomial approximation.

We can observe that the maximal value of the error in this case is less than 0.0008.

4. CONCLUSIONS

The examples presented here, with applications in nonlinear mechanics, show that our method performs well for equations such as the Duffing equation and Van der Pol equation.

A smaller value of the approximation error can be obtained by using a larger degree of the approximating polynomial. In this paper we used cubic and 5th degree, but if a higher precision is needed, polynomials of higher degree can be computed.

Moreover, this method can be easily extended for other types of equations and systems of equations, such as nonlinear systems, nonlinear differential systems, integral equations, and as such it can be considered a powerful tool for the computation of approximate solutions for nonlinear problems.

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