Remarks on Suzuki (C)-condition

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Abstract: In this manuscript, the results of Suzuki [Fixed point theorems and convergence theorems for some generalized non-expansive mappings, **340** (2009) 1088-1095] are considered and some new fixed point theorems are given.

Keywords: Contraction Mapping, Fixed Point Theory.

1. INTRODUCTION

The Banach contraction mapping principle (1) has a crucial role in Fixed Point Theory and has many applications in several branches of mathematics and also in economics. A self-mapping T on a metric space X is called contraction if for each $x, y \in X$, there exists a constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq d(x, y). \tag{1}$$

Due to Banach we know that every contraction on a complete metric space has a unique fixed point. This theorem, known as the Banach contraction mapping principle, is formulated in his thesis in 1920 and published in 1922 (1). After Banach, many authors attempt to generalize the Banach contraction mapping principle such as Kannan (6), Reich (7), Chatterjea (2), Hardy and Rogers (5), Ćirić (3) and many others.

Very recently, Suzuki proved the following fixed point theorem:

Theorem 1. (Suzuki (10).) Let (X, d) be a compact metric space and let T be a mapping on X. Assume $\frac{1}{2}d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Then T has a unique fixed point.

This result is based on the following two theorems:

Theorem 2. (Edelstein (4).) Let (X, d) be a compact metric space and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

Theorem 3. (Suzuki (9; 8).) Define a nonincreasing function θ from [0,1) onto (1/2,1] by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \leq r \leq 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases}$$

Then for a metric space (X, d), the following are equivalent:

(1) X is complete.
(2) Every mapping T on X satisfying the following has a fixed point: There exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$.

A mapping T on a subset K of a Banach space E is called a *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

Definition 4. ((9; 8)) Let T be a mapping on a subset K of a Banach space E. Then T is said to satisfy (C)-*condition* if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies that } \|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$.

Let $F(T)$ be the set of all fixed points of a mapping T. A mapping T on a subset K of a Banach space E is called a *quasi-nonexpansive mapping* if $\|Tx - z\| \leq \|x - z\|$ for all $x \in K$ and $z \in F(T)$.

We suggest a new definition which is a modification of Suzuki's C-condition:

Definition 5. Let T be a mapping on a subset K of a Banach space E. Then T is said to satisfy (for all $x, y \in K$)

(RSC) Reich-Suzuki-(C) condition (in short, (RSC)-condition) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies that}$$

$$\|Tx - Ty\| \leq \frac{1}{3}\Big[\|x - y\| + \|Tx - x\| + \|y - Ty\|\Big],$$

(RSCS) Reich-Chatterjea-Suzuki-(C) condition (in short, $(RCSC)$-condition) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies that}$$

$$\|Tx - Ty\| \leq \frac{1}{3}\Big[\|x - y\| + \|Tx - y\| + \|x - Ty\|\Big],$$

(HRSC) Hardy-Rogers-Suzuki-(C) condition (in short, $(HRSC)$-condition) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \text{ implies that} \|Tx - Ty\|$$

$$\leq \frac{1}{5}\Big[\|x - y\| + \|Tx - x\| + \|y - Ty\| + \|Tx - y\| + \|x - Ty\|\Big].$$

In this manuscript, we give new theorems that can be considered as a extension of the results of Suzuki (9) and also Singh-Mishra (12).

2. SOME BASIC OBSERVATIONS

Proposition 6. If a mapping T satisfies (RSC)-condition and has a fixed point, then it is quasi-nonexpansive mapping.

Proof. Let T be a mapping on a subset K of a Banach space E and satisfy (RSC)-condition. Suppose T has a fixed point, in other words, $z \in F(T)$. Hence, $0 = \frac{1}{2}\|z - Tz\| \leq \|z - y\|$ implies that

$$\begin{aligned}
&\|Tz - Ty\| \leq \frac{1}{3}\Big[\|z-y\| + \|Tz - z\| + \|y - Ty\|\Big] \\
&= \frac{1}{3}[\|z-y\| + \|y-Ty\|] \\
&\leq \frac{1}{3}\Big[\|z-y\| + \|y-z\| + \|z-Ty\|\Big] \\
&= \frac{1}{3}\Big[2\|y-z\| + \|Tz-Ty\|\Big]
\end{aligned} \tag{2}$$

thus $\|z - Ty\| = \|Tz - Ty\| \leq \|z - y\|$ which completes the proof.

Proposition 7. If a mapping T satisfies $(RCSC)$-condition and has a fixed point, then it is quasi-nonexpansive mapping.

Proof. Let T be a mapping on a subset K of a Banach space E and satisfy $(RCSC)$-condition. Suppose T has a fixed point, in other words, $z \in F(T)$. Clearly, $0 = \frac{1}{2}\|z - Tz\| \leq \|z - y\|$ implies that

$$\begin{aligned}
&\|Tz - Ty\| \leq \frac{1}{3}\Big[\|z-y\| + \|Tz - y\| + \|z - Ty\|\Big] \\
&= \frac{1}{3}\Big[2\|y-z\| + \|Tz-Ty\|\Big]
\end{aligned} \tag{3}$$

thus $\|z - Ty\| = \|Tz - Ty\| \leq \|z - y\|$ which completes the proof.

Proposition 8. If a mapping T satisfies $(HRSC)$-condition and has a fixed point, then it is quasi-nonexpansive mapping.

Proof. Let T be a mapping on a subset K of a Banach space E and satisfy $(HRSC)$-condition. Suppose T has a fixed point, in other words, $z \in F(T)$. Therefore, $0 = \frac{1}{2}\|z - Tz\| \leq \|z - y\|$ implies that

$$\begin{aligned}
&\|Tz - Ty\| \leq \frac{1}{5}\Big[\|z-y\| + \|Tz - z\| \\
&+\|y - Ty\| + \|Tz - y\| + \|z - Ty\|\Big] \\
&= \frac{1}{5}[2\|z-y\| + \|y-Ty\| + \|z-Ty\|] \\
&\leq \frac{1}{5}\Big[2\|z-y\| + \|y-z\| + \|z-Ty\| + \|z-Ty\|\Big] \\
&= \frac{1}{5}\Big[3\|y-z\| + 2\|Tz-Ty\|\Big]
\end{aligned} \tag{4}$$

thus $\|z - Ty\| = \|Tz - Ty\| \leq \|z - y\|$ which completes the proof.

Proposition 9. Let T be a mapping on a closed subset K of a Banach space E. Assume that T satisfies (RSC)-condition. Then $F(T)$ is closed. Moreover, E is strictly convex and K is convex, then $F(T)$ is also convex.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and converge to a point $x \in K$. It is clear that

$$\frac{1}{2}\|x_n - Tx_n\| = 0 \leq \|x_n - x\| \quad \text{for} n \in \mathbb{N}.$$

Thus, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \|x_n - Tx\| = \limsup_{n\to\infty} \|Tx_n - Tx\| \\
&\leq \limsup_{n\to\infty} \frac{1}{3}\Big[\|x_n - x\| + \|Tx_n - x_n\| + \|x - Tx\|\Big] \\
&= \limsup_{n\to\infty} \frac{1}{3}[\|x_n - x\| + \|x - Tx\|] \\
&\leq \limsup_{n\to\infty} \frac{1}{3}\Big[2\|x - x_n\| + \|x_n - Tx\|\Big].
\end{aligned} \tag{5}$$

Thus, we have

$$\frac{2}{3}\limsup_{n\to\infty} \|x_n - Tx\| \leq \frac{2}{3}\limsup_{n\to\infty} \|x - x_n\| = 0$$

which implies that $\{x_n\}$ converges to Tz. By uniqueness of limit, $Tz = z$. Hence, $z \in F(T)$, that is, $F(T)$ is closed.

Suppose that K is convex and E is strictly convex. Take $x, y \in F(T)$ with $x \neq y$ and define $z = tx + (1-t)y \in K$ for fixed $t \in (0, 1)$. Since E is strictly convex, then there exits $s \in [0, 1]$ such that $Tz = sx + (1-s)y$. Thus,

$$\begin{aligned}
(1-s)\|x-y\| &= \|Tx - Tz\| \\
\leq \frac{1}{3}[\|x-z\| &+ \|Tx - x\| + \|Tz - z\|] \\
&= \frac{1}{3}[\|x-z\| + \|Tz-z\|] \\
\leq \frac{1}{3}[\|x-z\| &+ \|Tz - Tx\| + \|Tx - z\|] \\
&= \frac{1}{3}[2\|x-z\| + \|Tz - Tx\|] \\
&= \frac{1}{3}[(1-s)\|x-y\| + 2(1-t)\|x-y\|]
\end{aligned} \tag{6}$$

and also

$$\begin{aligned}
s\|x-y\| &= \|Ty - Tz\| \\
\leq \frac{1}{3}[2\|y-z\| &+ \|Ty - y\| + \|Tz - z\|] \\
&= \frac{1}{3}[\|y-z\| + \|Tz-z\|] \\
\leq \frac{1}{3}[\|y-z\| &+ \|Tz - Ty\| + \|Ty - z\|] \\
&= \frac{1}{3}[2\|y-z\| + \|Tz - Ty\|] \\
&= \frac{1}{3}[s\|x-y\| + 2t\|x-y\|].
\end{aligned} \tag{7}$$

One conclude from (6) and (7) that $(1-s) \leq (1-t)$ and $s \leq t$, and hence $s = t$. Thus, $z \in F(T)$.

Proposition 10. Let T be a mapping on a closed subset K of a Banach space E. Assume that T satisfies $(RCSC)$-condition. Then $F(T)$ is closed. Moreover, E is strictly convex and K is convex, then $F(T)$ is also convex.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and converge to a point $x \in K$. It is clear that

$$\frac{1}{2}\|x_n - Tx_n\| = 0 \leq \|x_n - x\| \quad \text{for} \quad n \in \mathbb{N}$$

Thus, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\|x_n-Tx\| = \limsup_{n\to\infty}\|Tx_n-Tx\| \\
&\le \limsup_{n\to\infty}\frac{1}{3}\Bigg[\|x_n-x\|+\|Tx_n-x\|+\|x_n-Tx\|\Bigg] \qquad (8)\\
&= \limsup_{n\to\infty}\frac{1}{3}\Bigg[2\|x_n-x\|+\|x_n-Tx\|\Bigg].
\end{aligned}$$

Thus, we have

$$\frac{2}{3}\limsup_{n\to\infty}\|x_n-Tx\| \le \frac{2}{3}\limsup_{n\to\infty}\|x-x_n\| = 0$$

which implies that $\{x_n\}$ converges to Tz. By uniqueness of limit, $Tz = z$. Hence, $z \in F(T)$, that is, $F(T)$ is closed.

Suppose that K is convex and E is strictly convex. Take $x, y \in F(T)$ with $x \neq y$ and define $z = tx + (1-t)y \in K$ for fixed $t \in (0,1)$. Since E is strictly convex, then there exits $s \in [0,1]$ such that $Tz = sx + (1-s)y$. Thus,

$$\begin{aligned}
&(1-s)\|x-y\| = \|Tx-Tz\| \\
&\le \frac{1}{3}[\|x-z\|+\|Tx-z\|+\|Tz-x\|] \\
&= \frac{1}{3}[\|x-z\|+\|x-z\|+\|Tz-Tx\|] \qquad (9)\\
&= \frac{1}{3}[2\|x-z\|+\|Tz-Tx\|] \\
&= \frac{1}{3}[(1-s)\|x-y\|+2(1-t)\|x-y\|]
\end{aligned}$$

and also

$$\begin{aligned}
&s\|x-y\| = \|Ty-Tz\| \\
&\le \frac{1}{3}[\|y-z\|+\|Ty-z\|+\|Tz-y\|]| \\
&= \frac{1}{3}[2\|y-z\|+\|Tz-y\|] \qquad (10)\\
&= \frac{1}{3}[s\|x-y\|+2t\|x-y\|].
\end{aligned}$$

One conclude from (9) and (10) that $(1-s) \le (1-t)$ and $s \le t$, and hence $s = t$. Thus, $z \in F(T)$.

Proposition 11. Let T be a mapping on a closed subset K of a Banach space E. Assume that T satisfies $(HRSC)$-condition. Then $F(T)$ is closed. Moreover, E is strictly convex and K is convex, then $F(T)$ is also convex.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ and converge to a point $x \in K$. It is clear that

$$\frac{1}{2}\|x_n-Tx_n\| = 0 \le \|x_n-x\| \quad \text{for} \quad n \in \mathbb{N}$$

Thus, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\|x_n-Tx\| = \limsup_{n\to\infty}\|Tx_n-Tx\| \\
&\le \limsup_{n\to\infty}\frac{1}{5}\Bigg[\|x_n-x\|+\|Tx_n-x_n\| \\
&+\|x-Tx\|+\|Tx_n-x\|+\|x_n-Tx\|\Bigg] \\
&= \limsup_{n\to\infty}\frac{1}{5}\Bigg[2\|x_n-x\|+\|x-Tx\|+\|x_n-Tx\|\Bigg] \qquad (11)\\
&\le \limsup_{n\to\infty}\frac{1}{5}\Bigg[2\|x_n-x\|+\|x-x_n\| \\
&+\|x_n-Tx\|+\|x_n-Tx\|\Bigg] \\
&\le \limsup_{n\to\infty}\frac{1}{5}\Bigg[3\|x_n-x\|+\|x-x_n\|+2\|x_n-Tx\|\Bigg].
\end{aligned}$$

Thus, we have

$$\frac{3}{5}\limsup_{n\to\infty}\|x_n-Tx\| \le \frac{3}{5}\limsup_{n\to\infty}\|x-x_n\| = 0$$

which implies that $\{x_n\}$ converges to Tz. By uniqueness of limit, $Tz = z$. Hence, $z \in F(T)$, that is, $F(T)$ is closed.

Suppose that K is convex and E is strictly convex. Take $x, y \in F(T)$ with $x \neq y$ and define $z = tx + (1-t)y \in K$ for fixed $t \in (0,1)$. Since E is strictly convex, then there exits $s \in [0,1]$ such that $Tz = sx + (1-s)y$. Hence,

$$\begin{aligned}
(1-s)\|x-y\| &= \|Tx-Tz\| \\
&\le \frac{1}{5}[\|x-z\|+\|Tx-x\|+\|Tz-z\| \\
&\quad +\|Tx-z\|+\|Tz-x\|] \\
&= \frac{1}{5}[\|x-z\|+\|x-z\|+\|Tz-Tx\|] \\
&= \frac{1}{5}[2\|x-z\|+\|Tz-z\|+\|Tz-Tx\|] \qquad (12)\\
&\le \frac{1}{5}[2\|x-z\|+\|Tz-x\|+\|x-z\|+\|Tz-Tx\|] \\
&= \frac{1}{5}[3\|x-z\|+2\|Tz-Tx\|] \\
&= \frac{1}{5}[2(1-s)\|x-y\|+3(1-t)\|x-y\|]
\end{aligned}$$

and also

$$\begin{aligned}
s\|x-y\| &= \|Ty-Tz\| \\
&\le \frac{1}{5}[\|y-z\|+\|Ty-y\|+\|Tz-z\| \\
&\quad +\|Ty-z\|+\|Tz-y\|]| \\
&= \frac{1}{5}[2\|y-z\|+\|Tz-z\|+\|Tz-y\|] \\
&\le \frac{1}{5}[2\|y-z\|+\|Tz-y\|+\|y-z\|+\|Tz-y\|] \qquad (13)\\
&= \frac{1}{5}[3\|y-z\|+2\|Tz-y\|] \\
&= \frac{1}{5}[3\|y-z\|+2\|Tz-Ty\|] \\
&= \frac{1}{5}[2s\|x-y\|+3t\|x-y\|].
\end{aligned}$$

One conclude from (12) and (13) that $(1-s) \le (1-t)$ and $s \le t$, and hence $s = t$. Thus, $z \in F(T)$.

Proposition 12. Let T be a mapping on a closed subset K of a Banach space E. and satisfy (RSC)-condition. Then, for $x, y \in K$, the following hold:

(i) $\|Tx-T^2x\| \le \|x-Tx\|$

(ii) either $\frac{1}{2}\|x-Tx\| \le \|x-y\|$ or $\frac{1}{2}\|Tx-T^2x\| \le \|Tx-y\|$

(iii) either $\|Tx-Ty\| \le \frac{1}{3}\Bigg[\|x-y\|+\|Tx-x\|+\|Ty-y\|\Bigg]$ or $\|T^2x-Ty\| \le \frac{1}{3}\Bigg[\|Tx-y\|+\|T^2x-Tx\|+\|Ty-y\|\Bigg]$

Proof. The first statement follows from (RSC)-condition. Indeed, we always have $\frac{1}{2}\|x-Tx\| \le \|x-Tx\|$ which yields that

$$\begin{aligned}
\|Tx-T^2x\| &\le \frac{1}{3}\Bigg[\|x-Tx\|+\|Tx-x\|+\|T^2x-Tx\|\Bigg] \\
&\le \frac{1}{3}\Bigg[2\|Tx-x\|+\|T^2x-Tx\|\Bigg]
\end{aligned}$$

which implies (i).

It is clear that (iii) is consequence of (ii). To prove (ii), assume the contrary, that is,

$$\frac{1}{2}\|x-Tx\| > \|x-y\| \quad \text{and} \quad \frac{1}{2}\|Tx-T^2x\| > \|Tx-y\|$$

holds for all $x, y \in K$. Then by triangle inequality and (i), we have

$$\begin{aligned}\|x-Tx\| &\le \|x-y\| + \|y-Tx\| \\ &< \frac{1}{2}\|x-Tx\| + \frac{1}{2}\|Tx-T^2x\| \\ &\le \frac{1}{2}\|x-Tx\| + \frac{1}{2}\|x-Tx\| = \|x-Tx\|\end{aligned}$$

which is a contraction. Thus, we have (ii).

Proposition 13. Let T be a mapping on a closed subset K of a Banach space E. and satisfy $(RCSC)$-condition. Then, for $x, y \in K$, the following hold:

(i) $\|Tx-T^2x\| \le \|x-Tx\|$

(ii) either $\frac{1}{2}\|x-Tx\| \le \|x-y\|$ or $\frac{1}{2}\|Tx-T^2x\| \le \|Tx-y\|$

(iii) either $\|Tx-Ty\| \le \frac{1}{3}\Big[\|x-y\|+\|Tx-x\|+\|Ty-y\|\Big]$ or

$\|T^2x-Ty\| \le \frac{1}{3}\Big[\|Tx-y\|+\|T^2x-y\|+\|Ty-Tx\|\Big]$

Proof. The first statement follows from $(RCSC)$-condition. Indeed, we always have $\frac{1}{2}\|x-Tx\| \le \|x-Tx\|$ which yields that

$$\begin{aligned}\|Tx-T^2x\| &\le \frac{1}{3}\Big[\|x-Tx\|+\|Tx-Tx\|+\|T^2x-x\|\Big] \\ &= \frac{1}{3}\Big[\|x-Tx\|+\|T^2x-x\|\Big] \\ &\le \frac{1}{3}\Big[\|x-Tx\|+\|Tx-x\|+\|T^2x-Tx\|\Big] \\ &\le \frac{1}{3}\Big[2\|Tx-x\|+\|T^2x-Tx\|\Big]\end{aligned}$$

which implies (i).

It is clear that (iii) is consequence of (ii). To prove (ii), assume the contrary, that is,

$$\frac{1}{2}\|x-Tx\| > \|x-y\| \quad \text{and} \quad \frac{1}{2}\|Tx-T^2x\| > \|Tx-y\|$$

holds for all $x, y \in K$. Then by triangle inequality and (i), we have

$$\begin{aligned}\|x-Tx\| &\le \|x-y\| + \|y-Tx\| \\ &< \frac{1}{2}\|x-Tx\| + \frac{1}{2}\|Tx-T^2x\| \\ &\le \frac{1}{2}\|x-Tx\| + \frac{1}{2}\|x-Tx\| = \|x-Tx\|\end{aligned}$$

which is a contraction. Thus, we have (ii).

Proposition 14. Let T be a mapping on a closed subset K of a Banach space E. and satisfy $(HRSC)$-condition. Then, for $x, y \in K$, the following hold:

(i) $\|Tx-T^2x\| \le \|x-Tx\|$

(ii) either $\frac{1}{2}\|x-Tx\| \le \|x-y\|$ or $\frac{1}{2}\|Tx-T^2x\| \le \|Tx-y\|$

(iii) either $\|Tx-Ty\| \le \frac{1}{5}\Big[\|x-y\|+\|Tx-x\| + \|Ty-y\|+\|Tx-y\|+\|Ty-x\|\Big]$ or

$\|T^2x-Ty\| \le \frac{1}{5}\Big[\|Tx-y\|+\|T^2x-Tx\|+\|Ty-y\|+ \|T^2x-y\|+\|Ty-Tx\|\Big]$

Proof. The first statement follows from $(HRSC)$-condition. Indeed, we always have $\frac{1}{2}\|x-Tx\| \le \|x-Tx\|$ which yields that

$$\begin{aligned}\|Tx-T^2x\| &\le \frac{1}{5}\Big[\|x-Tx\|+\|Tx-x\| \\ &\quad +\|T^2x-Tx\|+\|Tx-Tx\|+\|T^2x-x\|\Big] \\ &= \frac{1}{5}\Big[2\|x-Tx\|+\|T^2x-Tx\|+\|T^2x-x\|\Big] \\ &\le \frac{1}{5}\Big[2\|x-Tx\|+\|T^2x-Tx\|+\|Tx-x\|+\|T^2x-Tx\|\Big] \\ &\le \frac{1}{5}\Big[3\|Tx-x\|+2\|T^2x-Tx\|\Big]\end{aligned}$$

which implies (i).

It is clear that (iii) is consequence of (ii). To prove (ii), assume the contrary, that is,

$$\frac{1}{2}\|x-Tx\| > \|x-y\| \quad \text{and} \quad \frac{1}{2}\|Tx-T^2x\| > \|Tx-y\|$$

holds for all $x, y \in K$. Then by triangle inequality and (i), we have

$$\begin{aligned}\|x-Tx\| &\le \|x-y\| + \|y-Tx\| \\ &< \frac{1}{2}\|x-Tx\| + \frac{1}{2}\|Tx-T^2x\| \\ &\le \frac{1}{2}\|x-Tx\| + \frac{1}{2}\|x-Tx\| = \|x-Tx\|\end{aligned}$$

which is a contraction. Thus, we have (ii).

3. MAIN RESULTS

Proposition 15. Let T be a mapping on a subset K of a Banach space E and satisfy (RSC)-condition. Then $\|x-Ty\| \le 7\|Tx-x\|+\|x-y\|$ holds for all $x, y \in K$.

Proof. Proof is based on Proposition 12 which says that either

$$\|Tx-Ty\| \le \|x-y\| \quad \text{or} \quad \|T^2x-Ty\| \le \|Tx-y\|$$

holds. Consider the first case, then we have

$$\begin{aligned}&\|x-Ty\| \le \|x-Tx\|+\|Tx-Ty\| \\ &\le \|x-Tx\| + \frac{1}{3}\{\|x-y\|+\|Tx-x\|+\|Ty-y\|\} \\ &\le \frac{4}{3}\|x-Tx\| + \frac{1}{3}\|x-y\| + \frac{1}{3}\|Ty-y\| \\ &\le \frac{4}{3}\|x-Tx\| + \frac{1}{3}\|x-y\| + \frac{1}{3}[\|Ty-x\|+\|x-y\|]\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{2}{3}\|x-Ty\| \le \frac{4}{3}\|x-Tx\| + \frac{2}{3}\|x-y\| \\ &\Leftrightarrow \|x-Ty\| \le 2\|x-Tx\| + \|x-y\|\end{aligned} \tag{14}$$

Take the second case into account. So we have

$$
\begin{aligned}
\|x-Ty\| &\le \|x-Tx\|+\|Tx-T^2x\|+\|T^2x-Ty\| \\
&\le 2\|x-Tx\|+\frac{1}{3}[\|Tx-y\|+\|T^2x-Tx\|+\|Ty-y\|] \\
&\le \frac{7}{3}\|x-Tx\|+\frac{1}{3}\|Tx-y\|+\frac{1}{3}\|Ty-y\| \\
&\le \frac{7}{3}\|x-Tx\|+\frac{1}{3}\|Tx-y\|+\frac{1}{3}[\|Ty-x\|+\|x-y\|]
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{3}\|x-Ty\| \le \frac{7}{3}\|x-Tx\|+\frac{1}{3}\|x-y\| \\
&\Leftrightarrow \|x-Ty\| \le 7\|x-Tx\|+\|x-y\|
\end{aligned} \tag{15}
$$

Hence, the result follows by (14) and (15).

Regarding the analogy, we omit the proof of the following Corollaries.

Corollary 16. Let T be a mapping on a subset K of a Banach space E and satisfy $(RCSC)$-condition. Then $\|x-Ty\| \le 9\|Tx-x\|+\|x-y\|$ holds for all $x, y \in K$.

Corollary 17. Let T be a mapping on a subset K of a Banach space E and satisfy $(HRSC)$-condition. Then $\|x-Ty\| \le 15\|Tx-x\|+\|x-y\|$ holds for all $x, y \in K$.

Theorem 18. Let T be a mapping on a compact convex subset K of a Banach space E and satisfies (RSC)-condition. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ holds. Then $\{x_n\}$ converge strongly to a fixed point of T.

Proof. Regarding K is compact, one can conclude that $\{x_n\}$ has an subsequence $\{x_{n_k}\}$ converges to some number, say z, in K By Proposition 15, we have

$$
\|x_{n_k}-Tz\| \le 7\|Tx_{n_k}-x_{n_k}\|+\|x_{n_k}-z\|, \ \forall k \in \mathbb{N}. \tag{16}
$$

Notice that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Taking into account of this fact with (16), we conclude that $\{x_{n_k}\}$ converges to Tz which implies that $Tz = z$. In other words, $z \in F(T)$. On account of the fact that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, we get

$$
\|x_{n+1}-z\| \le \lambda\|Tx_n-z\|+(1-\lambda)\|x_n-z\|
$$

for $n \in \mathbb{N}$. Thus, $\{x_n\}$ converges to z.

Corollary 19. Let T be a mapping on a compact convex subset K of a Banach space E. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ holds. If T satisfies one of the following:

(1) $(RCSC)$-condition,
(2) $(HRSC)$-condition,

then $\{x_n\}$ converge strongly to a fixed point of T.

Theorem 20. Let E be a Banach space and $T, S : K \to E$ such that $T(K) \subset S(K)$ and $S(K)$ is compact convex subset of E. Assume for $x, y \in K$,

$$
\begin{aligned}
&\frac{1}{2}\|Sx-Tx\| \le \|Sx-Sy\| \Rightarrow \|Tx-Ty\| \\
&\le \frac{1}{3}[\|Sx-Sy\|+\|Sx-Tx\|+\|Ty-Sy\|].
\end{aligned}
$$

Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0$ holds. Then T and S have a coincidence point.

Proof. Let $R : S(K) \to S(K)$ where $Ra = T(S^{-1}a)$ for each $a \in S(K)$. It is clear that R is well-defined. Indeed, take $x, y \in S^{-1}a$ such that $b = Tx$ and $c = Ty$. For $x \in S^{-1}a$ we obtain that $Ra = Tx$ and $Ra \subset S(K)$ since $T(K) \subset S(K)$. Thus, since $Sx = Sy$ we get $b = c$. Thus, R is well-defined.

Let $a, b \in S(K)$ such that $\frac{1}{2}\|a-Ra\| \le \|a-b\|$. Then for $x \in S^{-1}a$ and $y \in S^{-1}b$, one has

$$
\begin{aligned}
&\frac{1}{2}\|Sx-Tx\| = \frac{1}{2}\|a-Ra\| \le \|a-b\| = \|Sx-Sy\| \\
&\Rightarrow \|Tx-Ty\| \le \frac{1}{3}[\|Sx-Sy\|+\|Sx-Tx\|+\|Ty-Sy\|].
\end{aligned}
$$

Thus, $\frac{1}{2}\|a-Ra\| \le \|a-b\|$ implies that

$$
\|Ra-Rb\| \le \frac{1}{3}[\|a-b\|+\|Ra-b\|+\|Rb-a\|].
$$

Thus, all conditions of Theorem 18 are satisfied. Thus, R has a common fixed point, say t. Then for any $z \in S^{-1}t$, we have $Tz = Rt = t = Sz$. Hence, S, T have a coincidence point.

Corollary 21. Let E be a Banach space and $T, S : K \to E$ such that $T(K) \subset S(K)$ and $S(K)$ is compact convex subset of E. Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0$ holds. If S, T satisfy one of the following:

$\frac{1}{2}\|Sx-Tx\| \le \|Sx-Sy\|$ implies that

$$
\|Tx-Ty\| \le \frac{1}{2}[\|Sx-Ty\|+\|Tx-Sy\|],
$$

$\frac{1}{2}\|Sx-Tx\| \le \|Sx-Sy\|$ implies that

$$
\|Tx-Ty\| \le \frac{1}{3}\{[\|Sx-Sy\|+\|Sx-Ty\|+\|Tx-Sy\|]\},
$$

$\frac{1}{2}\|Sx-Tx\| \le \|Sx-Sy\|$ implies that

$$
\begin{aligned}
&\|Tx-Ty\| \le \frac{1}{5}\{[\|Sx-Sy\|+\|Sx-Tx\| \\
&+\|Ty-Sy\|+\|Sx-Ty\|+\|Tx-Sy\|]\},
\end{aligned}
$$

then T and S have a coincidence point.

Definition 22. Let E be a a Banach space. E is said to have Opial property (11) if for each weakly convergent sequence $\{x_n\}$ in E with weak limit z

$$
\liminf_{n\to\infty}\|x_n-z\| \le \liminf_{n\to\infty}\|x_n-y\|, \text{ for all } y \in E \text{ with } y \ne z.
$$

All Hilbert spaces, all finite dimensional Banach space and Banach sequence spaces $\ell_p (1 \le p < \infty)$ have the Opial property (See (9)).

Proposition 23. Let T be a mapping on a subset K of a Banach space E with Opial property and satisfy (RSC)-condition. If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$. That is $I - T$ is demiclosed at zero.

Proof. Due to Proposition 15, we have

$$
\|x_n-Tz\| \le 7\|Tx_n-x_n\|+\|x_n-z\|, \text{ for all } n \in \mathbb{N}.
$$

Hence,

$$\liminf_{n\to\infty}\|x_n - Tz\| \le \liminf_{n\to\infty}\|x_n - z\|.$$

Thus, Opial property implies that $Tz = z$.

Corollary 24. Let T be a mapping on a subset K of a Banach space E with Opial property and satisfy one of the following

(1) $(RSSC)$-condition,
(2) $(HRSC)$-condition.

If $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$. That is $I - T$ is demiclosed at zero.

Theorem 25. Let T be a mapping on a weakly compact convex subset K of a Banach space E with Opial property and satisfy (RSC)-condition. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ holds. Then $\{x_n\}$ converge weakly to a fixed point of T.

Proof. We have $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since K is weakly compact, one can conclude that $\{x_n\}$ has an subsequence $\{x_{n_k}\}$ which converges weakly to a number, say z, in E. On account of Proposition 23, we observe that z is a fixed point of T. Note that $\{\|x_n - z\|\}$ is a nondecreasing sequence. Indeed,

$$\|x_{n+1} - z\| \le \lambda\|Tx_n - z\| + (1-\lambda)\|x_n - z\|.$$

We show $\{x_n\}$ converges to z. Assume the contrary, that is, $\{x_n\}$ does not converge to z. Then there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ and $u \in K$ such that $\{x_{n_m}\}$ converges weakly to u and $u \ne z$. By Proposition 23 $Tu = u$. Since E has a Opial property,

$$\begin{aligned}&\lim_{n\to\infty}\|x_n - z\| = \lim_{k\to\infty}\|x_{n_k} - z\|\\ &< \lim_{k\to\infty}\|x_{n_k} - u\| = \lim_{n\to\infty}\|x_n - u\| = \lim_{m\to\infty}\|x_{n_m} - u\|\\ &< \lim_{m\to\infty}\|x_{n_m} - z\| = \lim_{n\to\infty}\|x_n - z\|\end{aligned} \tag{17}$$

which is a contradiction. Hence, proof is completed.

Corollary 26. Let T be a mapping on a weakly compact convex subset K of a Banach space E with Opial property and satisfy one of the following

(1) $(RSSC)$-condition,
(2) $(HRSC)$-condition.

Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ holds. Then $\{x_n\}$ converge weakly to a fixed point of T.

Theorem 27. Let E be a Banach space and $T, S : K \to E$ such that $T(K) \subset S(K)$ and $S(K)$ is weakly compact convex subset of E with Opial property. Assume for $x, y \in K$, then $\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\|$ implies that

$$\|Tx - Ty\| \le \frac{1}{3}[\|Sx - Sy\| + \|Tx - Sy\| + \|Sx - Ty\|].$$

Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0$ holds. Then T and S have a coincidence point.

Regarding the analogy with the proof of Theorem 20, we omit the proof.

Corollary 28. Let E be a Banach space and $T, S : K \to E$ such that $T(K) \subset S(K)$ and $S(K)$ is weakly compact convex subset of E with Opial property. Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - Sx_n\| = 0$ holds. If S, T satisfy one of the following:

(i) $\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\|$ implies that

$$\begin{aligned}\|Tx - Ty\| \le \frac{1}{2}\{[\|Sx - Sy\| + \|Sx - Ty\|\\ + \|Tx - Sy\|]\},\end{aligned} \tag{18}$$

(ii) $\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\|$ implies that

$$\begin{aligned}\|Tx - Ty\| \le \frac{1}{5}[\|Sx - Sy\| + \|Sx - Tx\|\\ + \|Ty - Sy\| + \|Sx - Ty\| + \|Tx - Sy\|],\end{aligned} \tag{19}$$

then T and S have a coincidence point.

A Banach space E is called *strictly convex* if $\|x + y\| < 2$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. A Banach space E is called *uniformly convex in every direction* (in short, UCED) if for $\varepsilon \in (0, 2]$ and $z \in E$ with $\|z\| = 1$, there exists $\delta := \delta(\varepsilon, z) > 0$ such that $\|x + y\| \le 2(1 - \delta)$ for all $x, y \in E$ with $\|x\| \le 1, \|y\| \le 1$ and $x - y \in \{tz : t \in [-2, -\varepsilon] \cup [\varepsilon, 2]\}$.

Lemma 29. (See (9)) For a Banach space E, the following are equivalent:

(1) E is UCED
(2) If sequence $\{u_n\}$ and $\{v_n\}$ in E satisfy $\lim_{n\to\infty}\|u_n\| = 1 = \lim_{n\to\infty}\|v_n\|$, $\lim_{n\to\infty}\|u_n + v_n\|$ and $\{u_n - v_n\} \subset \{tw : t \in \mathbb{R}\}$ for some $w \in E$ with $\|w\| = 1$, then $\lim_{n\to\infty}\|u_n - v_n\| = 0$ holds.

Lemma 30. (See (9)) For a Banach space E, the following are equivalent:

(1) E is UCED
(2) If $\{x_n\}$ is a bounded sequence in E, then a function f on E defined by $f(x) = \limsup_{n\to\infty}\|x_n - x\|$ is strictly quasi-convex, that is,
$$f(tx + (1-t)y) < \max\{f(x), f(y)\}$$
for all $t \in (0, 1)$ and $x, y \in E$ with $x \ne y$.

Theorem 31. Let T be a mapping on a weakly compact convex subset K of a UCED Banach space E and satisfy (RSC)-condition. Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$ holds. Then T has a fixed point theorem.

Proof. Set a sequence $\{x_n\}$ in K in a way that $x_{n+1} = \frac{1}{2}Tx_n + \frac{1}{2}x_n$ for each $n \in \mathbb{N}$ where $x_1 \in K$. Notice that $\limsup_{n\to\infty}\|Tx_n - x_n\| = 0$. Define a continuous convex function f from K into $[0, \infty)$ by $f(x) = \limsup_{n\to\infty}\|x_n - x\|$, for all $x \in K$. Since K is weakly compact and f is weakly lower semi-continuous, there exists $z \in K$ such that $f(z) = \min\{f(x) : x \in K\}$. Regarding Proposition 15, we have $\|x_n - Tz\| \le 7\|Tx_n - x_n\| + \|x_n - z\|$ and thus $f(Tz) \le f(z)$. On account of $f(z)$ is the minimum, $f(z) = f(Tz)$ holds. To show $Tz = z$ we assume the contrary, that is $Tz \ne z$. Since f is strictly quasi-convex, we have

$$f(z) \leq f(\frac{z+Tz}{2}) < \max\{f(z), f(Tz)\} = f(z)$$

which is a contradiction. Thus, we get the desired result.

Corollary 32. Let T be a mapping on a weakly compact convex subset K of a UCED Banach space E and satisfy one of the following

(1) $(RSSC)$-condition,
(2) $(HRSC)$-condition.

Define a sequence $\{x_n\}$ in K by $x_1 \in K$ and $x_{n+1} = \lambda T x_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where λ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ holds. Then T has a fixed point theorem.

Theorem 33. Let $\mathcal{S}$ be a family of commuting mappings on a weakly compact convex subset K of a Banach space E. Suppose each mapping in $\mathcal{S}$ satisfy (RSC)-condition. Then $\mathcal{S}$ has a common fixed point theorem.

Proof. Let $I = \{1, 2, ..., \nu\}$ be a index set. Let $T_i \in \mathcal{S}, i \in I$. Due to Theorem 31, T_i has a fixed point in K, that is, $F(T_i) \neq \emptyset$ for $i \in I$. Proposition 9 implies that each $F(T_i)$ is closed and convex. Suppose that $F := \cap_{i=1}^{k-1} F(T_i)$ is non-empty, closed and convex for some $k \in \mathbb{N}$ such that $1 < k \leq \nu$. For $x \in F$ and $i \in I$ with $1 \leq i < k$, $T_k x = T_k \circ T_i x = T_i \circ T_k x$ since S is commuting. Thus, $T_k x$ is a fixed point of T_i which yields $T_k x \in F$. So, $T_k(F) \subset F$. In other words, $T_k(F) \subset F$. By Theorem 31, T_k has a fixed point in F, that is, $F \cap F(T_k) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$.

Due to Proposition 9, this set is closed and convex. By induction, we obtain $\cap_{i=1}^{\nu} F(T_i) \neq \emptyset$. That is equivalent to say that $\{F(T) : T \in \mathcal{S}\}$ has the finite intersection property. Since K is weakly compact and $F(T)$ is weakly closed for every $T \in \mathcal{S}$, then $\cap_{T\in\mathcal{S}} F(T) \neq \emptyset$.

Corollary 34. Let $\mathcal{S}$ be a family of commuting mappings on a weakly compact convex subset K of a Banach space E. Suppose each mapping in $\mathcal{S}$ and satisfy one of the following

(1) $(RSSC)$-condition,
(2) $(HRSC)$-condition.

Then $\mathcal{S}$ has a common fixed point theorem.

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