# Calculation of a static potential created by plane fractal cluster

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**Abstract:** In this paper we demonstrate new approach that can help in calculation of electrostatic potential of a fractal (self-similar) cluster that is created by a system of charged particles. For this purpose we used the simplified model of a plane dendrite cluster (Feder 1988) that is generated by a system of the concentric charged rings located in some horizontal plane (see Fig.2.) The radiuses and charges of the system of concentric rings satisfy correspondingly to relationships: $r_n = r_0\xi^n$, and $e_n = e_0 b^n$, where *n* determines the number of the current ring. The self-similar structure of the system considered allows to reduce the problem to consideration of the functional equation that similar to the conventional scaling equation. Its solution represents itself the sum of power-low terms of integer order and non-integer power-law term multiplied to a log-periodic function (Nigmatullin 2005, Nigmatullin et al 2005). The appearance of this term was confirmed numerically for internal region of the self-similar cluster ($r_0 << r << r_{N-1}$, where $r_0, r_{N-1}$ determine the smallest and the largest radiuses of the rings correspondingly. The results were obtained for homogeneously ($b > 0$) and heterogeneously ($b < 0$) charged rings. We expect that this approach allows to consider more complex self-similar structures with different geometry.

Keywords**:** fractal (self-similar) charged clusters; electrostatic potential.

## 1. INTRODUCTION

To present time the influence of potential created by the static electric field on a geometric structure of the dendrite growth is intensively investigated (Kaply et al 2003, Mikhailov et al 1995, Smimov 1986, Smimov 1991) – formation of fractal threads and cluster-cluster aggregation under the applied electric field. (Vicsek, T 1992) – electrochemical deposition; but in solution of the inverse problem we found only a paper (Kornyshev 1991), where the influence of a self-similar structure on the generated potential was taken into account. The results presented in that paper are qualitative and not complete, because the discrete structure of the fractal considered plays an essential role. In this paper we calculated the electrostatic potential that is generated by a system including $N$ charged rings located in the horizontal plane (Fig. 2). This system was chosen because of one reason. It presents a plane model of dendrite cluster absorbed on a solid substrate. (Fig. 1). For this system one can show that the potential created in a certain region of space, which is determined as the internal region ($r_0 << r << r_{N-1}$, where $r_0, r_{N-1}$ – the smallest and the largest radiuses of the concentric rings correspondingly, $r$ is the current radius connecting the centre of the system with some point $P$ belonging to up or down half-space), satisfies to the scaling functional equation (Nigmatullin 2005) (see Appendix A). Its solution allows to find the nontrivial behavior in the region determined above. We mean the appearance of the term containing the non-integer power-law function combined with log-periodic term. Alongside with this basic result we present the results of numerical verification that confirm the basic positions of the theory.

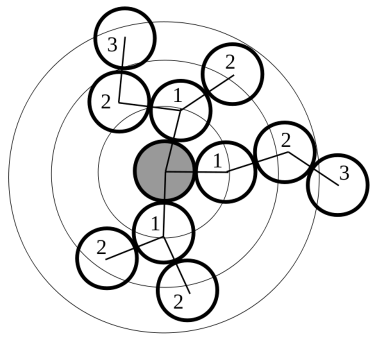



Fig. 1. On this figure we show schematically a fragment of a plane dendrite cluster and its fragmentation on a system of charged rings. This excess charge is created in the result of interaction of absorbed particles with substrate.

## 2. DESCRIPTION OF THE MODEL

Using the superposition principle for the electric field one can write the total potential as a sum of potentials created by its separate parts.

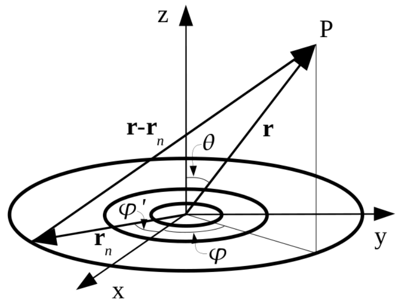



Fig. 2. On this figure we show the system of concentric rings which is used for description of the plane model of dendrite cluster. All rings are located in horizontal plane. Their sizes obey to expression $r_n = r_0 \xi^n$, with charge $e_n = e_0 b^n$; $n$ is the current number of a ring ($0 \le n \le N-1$), $r$ is a radius vector connecting a point P with the centre of the rings.

Let us determine a potential created in the point $P$ which is created by the $n$-th ring as $\Phi_n$, then the total potential in this point is expressed as

$$\Phi(\mathbf{r}) = \sum_{n=0}^{N-1} \Phi_n(\mathbf{r}). \quad (1)$$

It is easy to show (see Appendix A) that the contribution from the separate ring with number $n$ is determined by expression

$$\Phi_n(r,\theta) = \frac{2}{\pi} \frac{e_n F(k_n)}{\sqrt{r^2 + r_n^2 + 2 r r_n \sin\theta}},$$

where $F(k_n) = \int_0^{2\pi} dx \left(1 - k_n^2 \sin^2 x\right)^{-1/2}$ is the complete elliptic integral of the first type, $k_n^2 = 4 r r_n \sin\theta / (r^2 + r_n^2 + 2 r r_n \sin\theta)$; $r_n, e_n$ as before determine the radius and the charge of the $n$-th ring; $r$ is the value of the radius-vector connecting the point $P$ with the centre of the concentric rings. In general case all analytic calculations are finished at this point without additional assumptions related to the values $r_n$ and $e_n$. But if we make additional assumptions about the self-similar character related to these values then one can extract additional information about the self-similar character of expression (1). Let us suppose that these values satisfy to additional relationships $r_n = r_0 \xi^n, e_n = e_0 b^n$, where $\xi > 1, b > 0$ determine the scaling parameters which provide the increasing of the radius and the keeping of the charge sign within the system considered. It is easy to show (see Appendix A) that in this case for the total potential (1) the following functional equation is correct

$$\Phi(\xi r) = \frac{b}{\xi} \Phi(r) + f(r),$$

where the correction to this equation is determined as $f(r) = \Phi_0(\xi r) - \Phi_N(\xi r)$. For further calculations we introduce the dimensionless ratio $z$ and normalize the total charge to the unit value ($r = z r_{N-1}$, $r_{N-1} = 1$, $\sum_{n=0}^{N-1} e_n = 1$). This selection of variables is realized for conveniences of further calculations. After all routine transformations the basic functional equation accept the following form

$$\Phi(\xi z) = \frac{b}{\xi} \Phi(z) + f(z). \quad (2)$$

Unfortunately, at present time there is no analytical methods of solutions of the functional equations of the type (2) for an arbitrary analytical function $f(z)$. But if this function can be decomposed to some Taylor series then the unknown coefficients can be found by means of the uncertain coefficients method. So, for an arbitrary analytical function $f(z)$ presented in the form of the Taylor series ($n \ge 0$) or asymptotic ($n < 0$) series $f(z) = \sum_{n=-\infty}^{\infty} A_n z^n$ the solution of (2) is found and its solution has the form

$$\Phi(z) = z^{\nu} \pi_{\nu}(\ln z) + g(z), \quad (3)$$

where $\nu = \ln b / \ln \xi - 1$, $\pi_{\nu}(\ln z)$ is a log-periodic function i.e. $\pi_{\nu}(\ln z \pm \ln \xi) = \pi_{\nu}(\ln z)$; $g(z)$ determines the modified power-law series generated by the decomposition of $f(z)$, in general case $g(z) = \sum_{n=-\infty}^{\infty} C_n z^n$.

The convergence of $f(z)$ depends on the region where the equation (2) is considered. We consider three regions

1) $r << r_0$ ($z << \xi^{-N+1}$); - the region located inside the smallest ring;

2) $r_0 << r << r_{N-1}$ ($\xi^{-N+1} << z << 1$); - internal cluster region

3) $r >> r_{N-1}$ ($z >> 1$); external region out of the cluster.

The most interesting result is obtained for the region (2), where the influence of the non-integer power-law term figuring in (3) is essential. For other regions (1) and (3) the influence of this term is negligible. The reasons of this behavior are discussed below.

In conclusion of this part we give the solution of equation (2) for an arbitrary analytical function $f(z)$. The substitution of (3) in equation (2) gives the following values for coefficients $C_n = \xi A_n / (\xi^{n+1} - b)$ which are found by the method of uncertain coefficients. From (2) it follows that at $b=\xi^n$ the power-law exponent $\nu$ becomes integer and the coefficient $C_n$ goes to infinity. So, for the partial case $\Phi(\xi z)=\xi^{n-1}\Phi(z)+A_{n-1}z^{n-1}$ (as $f(z)$ the power-law function is used) the method of uncertain coefficients does give a chance to find the desired function but nevertheless one can sort out $g(z)$ in the following form

$$g(z)=\frac{A_{n-1}}{\xi^{n-1}\ln\xi}z^{n-1}\ln z\,.$$

Using the general solution one can verify all analytical conclusions numerically.

## 3. NUMERICAL TESTS

The analytical dependencies were checked numerically for the interval $[\xi^{-N+2};\xi^{-1}]$ on the segment representing the part of the radius-vector $r$. This interval was divided (in logarithmic scale) on $10^3$ equal segments and for each segment the mean point $z_i$ was chosen. For each point independently the values of potentials $\Phi(z_i)$ and function $g(z_i)$ were calculated (see Figs. 3 a, b) with the given accuracy (the absolute values of the error does not exceed the value $\varepsilon=10^{-12}$). Realizing the difference $\Phi(z_i)-g(z_i)$ one can get the non-integer power-law term in explicit form (Figs. 3 c, d). For verification of the power-law dependence it was presented in the double-log scale. For the case $b<0$ the constant term in the function $\pi_\nu(\ln z)$ close to zero and the function becomes alternating-sign; for this case the logarithmic scale was used only for the abscissa. To see the oscillating part the difference $\Phi(z_i)-g(z_i)$ was divided on $z^\nu$ then the mean value was subtracted and the remnant was constructed in logarithmic scale (Figs. 3 e, f). For clarity some curves are shifted with respect to each other. In all cases we confirmed the existence of the power-law term together with oscillating part in the potential considered. It is important to notice here that these oscillations are not appeared in the result of subtraction of the series $g(z)$; the same results are appeared when we put $g(z)$ to the constant value. In this case the zero approximation for $g(z)$ serves the value $g(z)=b^N e_0/(\xi-b)$, but in this case we observe the narrowing effect for the region, where the effect of oscillations of the log-periodic function $\pi_\nu(\ln z)$ is observed. This effect influences on accuracy of calculation of the fitting parameters of this function.

The separate interest is related with the fitting of the log-periodic function $\pi_\nu(\ln z)$ for the testing of the suggested approach and determination of the basic parameters if they are not known. In paper (Nigmatullin 2005) it was shown that for the fifing of this function one can use the segment of the log-periodic Fourier series, which includes the first leading

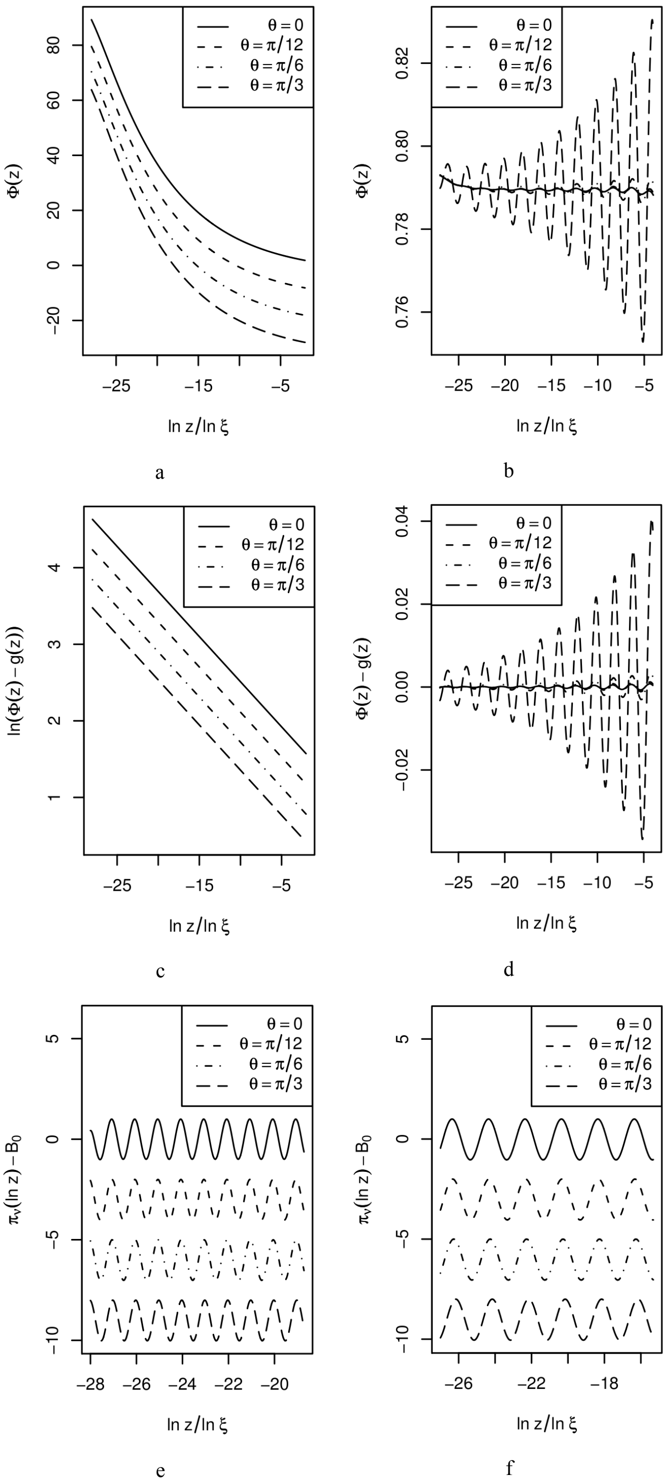
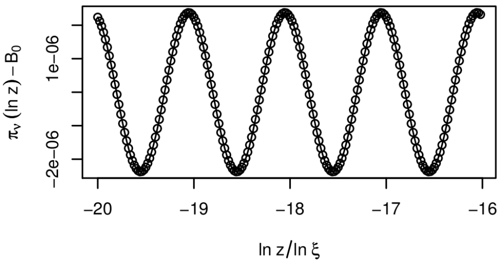


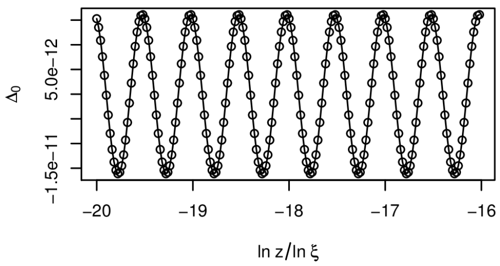

Fig. 3. On this figure we show the potential created by the system of the charged concentric rings a) $\xi=1.8$, $b=1.6$, $N=30$; b) $\xi=1.8$, $b=-2$, $N=31$; c), d) fractional power-law term for the same set of the parameters for the alternating-sign case at $|b|>\xi$ the increasing of amplitudes is observed.; e), f) the normalized log-periodic functions obtained for the same values of the parameters. The curves presented on figures a), c), e), f), are shifted relatively the vertical axes with respect to each other for clarity.

terms. In our case we know the leading frequency mode equaled $2\pi / \ln\xi$ ), so the unknown amplitudes can be found by means of the linear least square method (LLSM). This approach helps to separate the contributions for each mode divisible to the basic one.

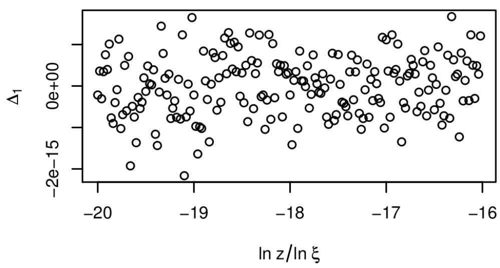
From Fig.4 one can notice the excellent agreement of the theory with numerical tests. One can fit (with the high accuracy) not only the basic mode. This result is related with the limits of accuracy which did not exceed the value $10^{-16}$. It is interesting to evaluate the contribution of these terms in the behaviour of the potential.




a



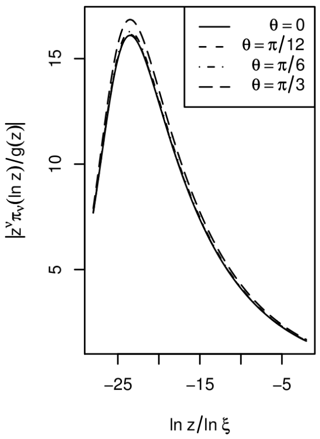

b



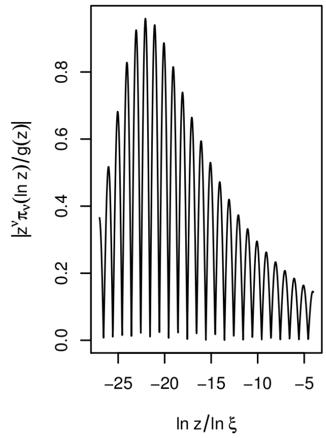

c

Fig. 4. On this figure we show: a) the fitting of the log-periodic function with the help of the leading mode which is expressed as $B_1 \cos(2\pi \ln z / \ln\xi + \alpha_1)$ ; b) the fitting of the difference between log-periodic function and the leading mode. One can fit the remnants $\Delta_0$ by the second mode which is divisible to the basic one. The fit was realized by the term $B_2 \cos(4\pi \ln z / \ln\xi + \alpha_2)$ ; c) after subtraction of these two terms only uniform noise is remained.

Fig.5 shows that the contribution of the term $z^\nu \pi_\nu(\ln z)$ can be essential. It is important to notice that this contribution has the principal character because the term containing $z^\nu$ cannot be obtained from the conventional decomposition of the Taylor series. This term is the basic consequence of *self-similar and discrete* character of the charge distribution in the system considered.




a




b

Fig. 5. On this figure we demonstrate the contribution of the non-integer term in the form of the ratio $\left| z^\nu \pi_\nu(\ln z) / g(z) \right|$ a) at $b > 0$ for some values of the angle $\theta$, b) at $b < 0$ for $\theta = \pi / 3$ the contribution becomes essential in spite of the value of small amplitude of the function $\pi_\nu(\ln z)$. In the second case (b) the potential has the oscillating character similar to $\pi_\nu(\ln z)$, but the envelope is similar to the behavior of the potential depicted on the left-hand side.

Let us consider the calculation of the potential created out of the cluster boundaries i.e at $z > 1$. One can show that in the vicinities of the unit value the function $g(z)$ has a jump (see Appendix B).

The numerical evaluations show that the equality $\Delta_1 = \Delta_2$ is satisfied with high accuracy for any admissible angle $\theta$. It means that the term $z^\nu \pi_\nu(\ln z)$ is "locked" within the sphere of the unit radius This unexpected result can evoke misunderstanding but it follows from the simplifications when a plane dendrite system is replaced by the set of rings If one replaces a set of circles by a circular layers (by analogy with spherical layers) of the width $\Delta r_n = \Delta r_0 \xi^n$ at condition $\Delta r_0 / r_0 << 1$ (Fig. 7 a) then the transition region with the width $\Delta r_0 \xi^{N-1}$ is appeared. It is easy to understand when the circular layer is divided on $m >> 1$ more thin layers that are approximately replaced by a set of circles. In the result of such division we obtain $m$ enclosed systems having $N$ circles. For each circle we have jump-like behavior of $g(z)$ at certain value of $z$, but this values will be uniformly distributed within the interval $[1 - \Delta r_0 / r_0 ; 1]$. In the limit $m \to \infty$ we obtain a smooth decreasing of the term $z^{\nu}\pi_{\nu}(\ln z)$ and the jump is replaced by a smooth transition region.

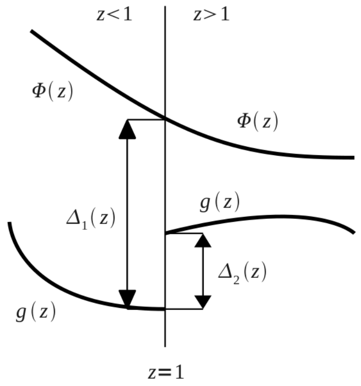



Fig. 6. On this figure we show the situation of the jump-like behavior for the function $g(z)$ when it passes through the value $z=1$. The values characterizing the jumps $\Delta_1$ and $\Delta_2$ for any admissible value of the angle $\theta$ with the high accuracy are equalled to each other that lead to the "locking" of the function $z^{\nu}\pi_{\nu}(\ln z)$ within the sphere of the unit radius. This effect is the consequence of simplifications that were initially introduced in consideration of this self-similar structure modelling the dendrite plane cluster.

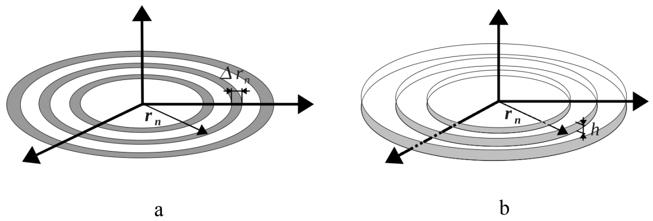



Fig. 7. On these figures we show possible modifications of replacement the system of the infinitely thin rings a) by a system of circle layers with the given width b) by a system of cylindrical circles having the given height.

Similar situation arises when we replace a set of rings by cylinder surfaces having the height $h << r_0$ (Fig. 7 b), in this case the width of the transition region will be equaled $\sqrt{(1-\cos\theta)^2 + h/r_{N-1}(2\cos\theta - h/r_{N-1})} - (1-\cos\theta)$. It is obvious that for real systems we should observe the smoothed transition region connecting two previous regions (1) and (2).

## 4. CONCLUSIONS

The basic conclusion can be formulated as follows: the *self-similar* (*fractal*) distribution of charges generates a potential having non-integer power-law behavior with oscillating log-periodic part. This term in admissible region can increase at condition ($|b| > \xi$). This result has the principal value because it does not follow from the conventional speculations and follows from the self-similar structure of the system considered. The system of plane self-similar rings models a dendrite cluster located on the plane surface and reveals some specific details in interaction of the charged particles with self-similar objects of such type. Definitely, the investigation of other self-similar geometries that can generate of potentials with this specific behavior will be continued.

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## Appendix A.

We choose on the ring with the fixed number $n$ an infinitesimal element of the arc with length $dl = rd\varphi'$. This selected element has a charge and gives a contribution to the potential created by a single ring. It is supposed that the charge distribution density is uniform for each ring. Summarizing all distributions from elementary arcs we obtain

$$\Phi_n(r,\theta,\varphi) = \frac{e_n}{2\pi}\int_0^{2\pi}\frac{d\varphi'}{\sqrt{r^2 + r_n^2 + 2rr_n\sin\theta\cos(\varphi-\varphi')}}. \quad (1a)$$

After integration of this expression over azimuth angles we have

$$\Phi_n(r,\theta) = \frac{2}{\pi}\frac{e_nF(k_n)}{\sqrt{r^2 + r_n^2 + 2rr_n\sin\theta}},$$

where $F(k_n)=\int_0^{2\pi} dx\left(1-k_n^2\sin^2 x\right)^{-1/2}$ is the complete elliptic integral of the first type with the argument $k_n^2=4rr_n\sin\theta/(r^2+r_n^2+2rr_n\sin\theta)$; $r_n, e_n$ determine the radius and charge of the $n$–th ring, $r$ –the value of the radius vector , connecting point $P$ with the centre of all rings.

Then we consider $\Phi(\xi r)=\sum_{n=0}^{N-1}\Phi_n(\xi r)$. The angle $\theta$ is omitted because it has the fixed value. This sum we present in the form

$$\Phi(\xi r)=\Phi_0(\xi r)+\sum_{n=1}^{N-1}\frac{e_n F(k_n)}{\sqrt{\xi^2 r^2+r_n^2+2\xi r r_n\sin\theta}}.$$

Then we bring out from the sum two values: $\xi$ from the square root and $b$ from the numerator of this expression.

$$\Phi(\xi r)=\Phi_0(\xi r)+\frac{b}{\xi}\sum_{n=1}^{N-1}\frac{e_{n-1}F(k_{n-1})}{\sqrt{r^2+r_{n-1}^2+2rr_{n-1}\sin\theta}}.$$

Then the limits of summation are decreased on the unit value

$$\Phi(\xi r)=\Phi_0(\xi r)+\frac{b}{\xi}\sum_{n=0}^{N-2}\frac{e_n F(k_n)}{\sqrt{r^2+r_n^2+2rr_n\sin\theta}}.$$

In the sum we obtain the expression, which is close to the initial one. In order to keep the same structure we add the term with $N-1$ and subtract it from the previous expression. In the result of this transformation we obtain

$$\Phi(\xi r)=\Phi_0(\xi r)+\frac{b}{\xi}\Phi(r)-\frac{b}{\xi}\frac{e_{N-1}F(k_{N-1})}{\sqrt{r^2+r_{N-1}^2+2rr_{N-1}\sin\theta}}.$$

The last term coincides with $\Phi_N(\xi r)$. Finally we obtain the equation

$$\Phi(\xi r)=\frac{b}{\xi}\Phi(r)+\Phi_0(\xi r)-\Phi_N(\xi r).$$

The difference $\Phi_0(\xi r)-\Phi_N(\xi r)$ we define through out the text as $f(r)$. For the alternating-sign case ($b<0$) we have for $f(r)$ the following expression

$$f(r)=\Phi_0(\xi^2 r)-\Phi_N(\xi^2 r)+\Phi_1(\xi^2 r)-\Phi_{N+1}(\xi^2 r).$$

The we decompose $f(z)$ into the infinite series, where $z$ is dimensionless and normalized radius ($r=zr_{N-1}$, $r_{N-1}=1$). The function $f(z)$ is formed by a pair of expressions of the type (1a). For their decomposition we use a small trick. Initially we decompose their integrands with the help of the Legendre polynomials and then integrate them. In the result of these transformations for the region (2) we obtain the following expression

$$f(z)=\frac{e_0}{r_0}\sum_{n=0}^{\infty}P_n\frac{1}{(\xi^N z)^{n+1}}-\frac{e_0 b^N}{\xi}\sum_{n=0}^{\infty}P_n z^n,$$

where the coefficients $P_n=P_n(\theta)=\left\langle P_n(\theta,\varphi-\varphi')\right\rangle_{\varphi-\varphi'}$ determine the integrated Legendre polynomials, $P_0=1$. For other regions these transformations are similar.

## Appendix B.

One can show that the function $g(z)$ at $z=1$ has the jump-like behaviour. For this purpose we consider the difference $\Delta_2=g(z)\big|_{z\to 1+0}-g(z)\big|_{z\to 1-0}$. Each term in the right-hand side has a separate integer series. After some simple transformations we obtain the following value for the jump.

$$\Delta_2=e_0 b^N\sum_{n=0}^{\infty}P_n\left[\frac{1}{\xi^{-n}-b}-\frac{1}{\xi^{n+1}-b}\right].$$

This expression does not converge to zero value. It coincides with the value $\Delta_1=\pi_\nu(\ln z)\big|_{z\to 1-0}$, which is obtained after the fitting of this function in the region (2).