Dynamical Symmetries of Second Order ODE

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Abstract: There is a class of dynamic symmetries which have invariants providing the lowering of the ODE order and containing the whole class of point symmetries. The procedure of the finding of dynamic symmetries for ODE of second order is suggested. Some examples of the usage of dynamic symmetries are given. The application of the obtained solutions to investigation of the nonlinear heat conduction equation is considered.

Keywords: ordinary differential equations, dynamical symmetries, Abel's equation, nonlinear heat conduction equation.

1. INTRODUCTION

The concept of dynamic symmetry is given, for example, [Stephani (1989)]. The differential equation, in this case, is replaced by ODE system of the first order:

$$y'' = f(x, y, y') \;\Leftrightarrow\; \frac{dy}{dx} = z, \;\; \frac{dz}{dx} = f(x, y, z)\,. \tag{1}$$

Then it is possible to consider the question of an infinitesimal transformation

$$X = \xi(x, y, z)\frac{\partial}{\partial x} + \eta(x, y, z)\frac{\partial}{\partial y} + \mu(x, y, z)\frac{\partial}{\partial z}, \tag{2}$$

transforming a solution of the system (1) again into the solution of the same system. Thus the operator (2) should satisfy the condition

$$[X, A] = \lambda(x, y, z)\, A, \tag{3}$$

where $A = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z}$.

It is important to note that the components of the operator (2) do not satisfy the prolongation formula suggested by S.Lie

$$\mu = \frac{d\eta}{dx} - z\frac{d\xi}{dx}\;\; .$$

The components $\xi\,, \eta\,, \mu$ of the dynamic symmetries operator X are defined only by the condition (3).

On application of point symmetries of ODEs the invariants are constructed with the found operator. These invariants are applied for the lowering of the original ODE order. For this aim it is necessary to solve ODEs of the first order. It is known that in some cases the problem of the invariants finding is equivalent to the problem of the original DE integration . In the case of dynamic symmetries in the general case it would be necessary to solve a dynamic system.

In the work [Timoshin (2009)] it is suggested to begin the procedure of the symmetries finding with the invariants, where the components of the relevant operator can be written out directly with the help of differentiation and arithmetical operations. The disadvantage of such an approach consists in the fact that the linear in the classic case defining system of DEs becomes nonlinear. However such an approach in the case of dynamic symmetries allows to take care beforehand about properties of the searched symmetries. In particular it is naturally to require that if the functions $u = u(x, y, y')\,, \; v = v(x, y, y')$ are the invariants of the operator (2), the expression

$$\frac{du}{dv} = \frac{u'_x + u'_y y' + u'_{y'} y''}{v'_x + v'_y y' + v'_{y'} y''}$$

also would be invariant of once prolonged dynamic symmetry (2). It is more convenient to write the components of the dynamic symmetry operator in the form

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial y'} + \mu_1(x, y, y', y'')\frac{\partial}{\partial y'}, \tag{4}$$

and to determine them by solving the system of equations

$$X\,u = 0, \quad X\,v = 0, \quad X\,\frac{du}{dv} = 0\,. \tag{5}$$

At the finding of the symmetries for ODE of the second order

$$F(x, y, y', y'') = 0,$$

the invariance criterion

$$XF\,|_{F=0} \equiv 0, \tag{6}$$

can be used. In the general case this criterion (6), as well as the invariance criterion (3) does not allow the splitting. The advantage of invariance criterion (6) consists in the fact that it allows saying about the dynamic symmetry (4) satisfying the equations (5) up to the constant functional factor.

Note, that all point symmetries can be considered as ansatze of dynamic ones when

$$u = \frac{\beta'_x + \beta'_y y'}{\alpha'_x + \alpha'_y y'}, \quad v = \beta(x, y)\,,$$

where $\alpha(x, y)\,, \beta(x, y)$ are arbitrary functions.

Thereupon with intending to preserve the splitting property for the criterion (6), it is naturally to state the question about dynamic symmetries as the question about the extension of the point symmetries set restricting oneself by the functions of two variables.

Consider now the question about dynamic symmetries spanned on three functions of two variables containing the whole set of point symmetries. First of all note the point symmetry operator

$$X=\xi\left(x,y\right)\frac{\partial}{\partial x}+\eta\left(x,y\right)\frac{\partial}{\partial y},$$

having the invariant $v=\tau\left(x,y\right)$, in variables $\left(v,y\right)$, takes the form

$$X=\chi\left(v,y\right)\frac{\partial}{\partial y}.$$

Representing the function $\chi\left(v,y\right)$ in the form

$$\chi\left(v,y\right)=\frac{1}{\alpha'_y},$$

the first differential invariant can be written out

$$u=\alpha'_v+\alpha'_y\frac{dy}{dv}.$$

Thus taking the first differential invariant in the form

$$u=\alpha+\beta\frac{dy}{dv},$$

it is possible to extend the point symmetry set with the help of three functions $\tau\left(x,y\right),\alpha\left(v,y\right),\beta\left(v,y\right)$.

The practical finding of the considered type of dynamic symmetries is convenient to carry out in the several steps:

- A Making the point variables change $t=\tau\left(x,y\right),y=y$, it is necessary to pass from the equation
$$F\left(t,y,\frac{dy}{dt},\frac{d^2y}{dt^2}\right)=0$$
to the equation
$$\Phi\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2}\right)=0.$$
- B On the invariants
$$v=x,\ \ u=\alpha\left(x,y\right)+\beta\left(x,y\right)\frac{dy}{dx},$$
solving the system (5), to construct the dynamic symmetry operator.
- C Splitting by y' the invariance criterion
$$X\Phi\left|_{\Phi=0}\right.\equiv 0$$
to write the determining system of differential equations.
- D To resolve the determining system.

In the work [Timoshin (2009)], by the example of the equation

$$y''=y'+f\left(y\right), \tag{7}$$

the realization of B, C, D steps is given. It is shown that the ODE (7) allows the dynamic symmetries, when the function $f\left(y\right)$ has the form

$$f\left(y\right)=Ay+\frac{B}{y}-\frac{B^2}{y^3}.$$

Explicit solutions of corresponding equations are given.

In the following section by the example of (7) the realization of all four steps is considered.

2. EXAMPLE OF DYNAMIC SYMMETRIES FINDING FOR ODE OF SECOND ORDER

Take the equation (7) in the form

$$\frac{d^2y}{dt^2}=\frac{dy}{dt}+f(y).$$

Making the point change of variables $t=\tau(x,y)$, come to the differential equation

$$\begin{gathered}\tau'_x y''-f(y){\tau'_x}^3-\left(\tau''_{xx}+{\tau'_x}^2+3f(y){\tau'_x}^2\tau'_y\right)y'-\\-\left(2\tau''_{xy}+2\tau'_x\tau'_y+3\tau'_x{\tau'_y}^2f(y)\right){y'}^2-\\-\left(\tau''_{yy}+{\tau'_y}^2+f(y){\tau'_y}^3\right){y'}^3=0.\end{gathered}\tag{8}$$

With the invariants

$$\begin{gathered}v=x,\quad u=\alpha(x,y)+\beta(x,y)y',\\ \frac{du}{dv}=\alpha'_x+\left(\beta'_x+\alpha'_y\right)y'+\beta'_y{y'}^2+\beta y''\end{gathered}$$

construct the dynamic symmetry operator (4) with components

$$\begin{gathered}\xi(x,y,y')=0,\ \eta(x,y,y')=1,\ \mu(x,y,y')=-\frac{\alpha'_y+\beta'_y y'}{\beta},\\ \mu_1(x,y,y',y'')=-\frac{\beta'_y}{\beta}y''+\frac{-\beta''_{yy}\beta+2{\beta'_y}^2}{\beta^2}{y'}^2+\\+\frac{-\alpha''_{yy}\beta+3\alpha'_y\beta'_y+\beta'_x\beta'_y-\beta''_{xy}\beta}{\beta^2}y'+\\+\frac{-\alpha''_{xy}\beta+\beta'_x\alpha'_y+{\alpha'_y}^2}{\beta^2}.\end{gathered}$$

Splitting the invariance criterion $X\Phi\left|_{\Phi=0}\right.\equiv 0$ by y', get the determining system of differential equations

$$\begin{gathered}-\beta^2\left(\tau'''_{xxy}\tau'_x+\tau''_{xy}{\tau'_x}^2+3f'(y){\tau'_x}^3\tau'_y+3f(y){\tau'_x}^2\tau'_y\tau''_{xy}+\right.\\\left.+3f(y){\tau'_x}^3\tau''_{yy}-\tau''_{xy}\tau''_{xx}\right)+4\alpha'_y\beta\tau'_x\tau''_{xy}+{\tau'_x}^2\left(4\alpha'_y\beta\tau'_y+\right.\\\left.+6\alpha'_y\beta f(y){\tau'_y}^2-\alpha''_{yy}\beta+3\beta'_y\alpha'_y-\beta''_{xy}\beta+\beta'_x\beta'_y\right)=0,\end{gathered}$$

$$\begin{gathered}\beta^2\left(-2\tau'''_{xyy}\tau'_x-2\tau''_{yy}{\tau'_x}^2-3f'(y){\tau'_x}^2{\tau'_y}^2-6f(y){\tau'_x}^2\tau'_y\tau''_{yy}+\right.\\\left.2\,{\tau''_{xy}}^2\right)+\beta\tau'_x\left(3\alpha'_y\tau''_{yy}+3\alpha'_y{\tau'_y}^2+3\alpha'_yf(y){\tau'_y}^3+2\beta'_y\tau''_{xy}\right)\\+{\tau'_x}^2\left(2\beta\beta'_y\tau'_y+3f(y)\beta\beta'_y{\tau'_y}^2-\beta''_{yy}\beta+2{\beta'_y}^2\right)=0,\end{gathered}$$

$$\begin{gathered}\beta\left(\tau''_{xy}\tau''_{yy}+\tau''_{xy}{\tau'_y}^2+\tau''_{xy}f(y){\tau'_y}^3-\tau'''_{yyy}\tau'_x-\right.\\\left.-2\tau'_x\tau'_y\tau''_{yy}-f'(y){\tau'_y}^3\tau'_x-3f(y)\tau''_{yy}\tau'_x{\tau'_y}^2\right)+\\+2\tau'_x\beta'_y\left(\tau''_{yy}+{\tau'_y}^2+f(y){\tau'_y}^3\right)=0,\end{gathered}$$

$$\begin{gathered}-\beta^2{\tau'_x}^3f'(y)-2{\tau'_x}^2\beta^2f(y)\tau''_{xy}+\alpha'_y\beta\tau''_{xx}+\\+\alpha'_y\beta{\tau'_x}^2+3\alpha'_y\beta{\tau'_x}^2f(y)\tau'_y-f(y)\beta\beta'_y{\tau'_x}^3-\\-\alpha''_{xy}\beta\tau'_x+\alpha'_y\beta'_x\tau'_x+\tau'_x{\alpha'_y}^2=0,\end{gathered}$$

which is the PDEs system of four equations relative to three unknown functions $\alpha(x,y),\beta(x,y),\tau(x,y)$.

We will try solutions of this system with the group analysis methods for PDEs, presented for example in [Ibragimov (1985), Ovsjannikov (1982)]. Considering the determining system as the system relative to the function $\tau(x,y)$, see that it has the symmetry

$$X=\frac{\partial}{\partial x}+a\frac{\partial}{\partial \tau}\,,$$

when the functions $\alpha(x,y),\beta(x,y)$ take the form

$$\alpha(x,y)=\alpha_1(x)+\alpha_2(y)\beta_1(x),\;\;\beta(x,y)=\beta_1(x)\beta_2(y).$$

The obtained symmetry prescribes the finding of the invariant solution in the form

$$\tau(x,y)=ax+\tau_1(y).$$

Using the written representations for the functions $\tau(x,y)$, $\alpha(x,y)$, $\beta(x,y)$, transform the determining system to the form

$$\begin{gathered}
-3a{\beta_2}^2f'(y){\tau_1'}^2+3\beta_2f(y)\tau_1'\left(a\beta_2'\tau_1'-2a\beta_2\tau_1''+\alpha_2'{\tau_1'}^2\right)+\\
+2a\left({\beta_2'}^2-{\beta_2}^2\tau_1''+\beta_2\beta_2'\tau_1'\right)+\\
+\beta_2\left(3\alpha_2'\left({\tau_1'}^2+\tau_1''\right)-a\beta_2''\right)=0,
\end{gathered}$$

$$\begin{gathered}
-\beta_2f'(y){\tau_1'}^3+f(y){\tau_1'}^2\left(2\beta_2'\tau_1'-3\beta_2\tau_1''\right)-\\
-\beta_2\tau_1'''-2\beta_2\tau_1'\tau_1''+2\beta_2'{\tau_1'}^2+2\beta_2'\tau_1''=0,
\end{gathered}$$

$$\begin{gathered}
-3a{\beta_2}^2f'(y)\tau_1'-3\beta_2f(y)\left(a\beta_2\tau_1''-2\alpha_2'{\tau_1'}^2\right)-\\
-\left(\alpha_2''\beta_2-3\alpha_2'\beta_2'-4\alpha_2'\tau_1'\beta_2\right)=0,
\end{gathered}$$

$$a^2{\beta_2}^2f'(y)+a\beta_2f(y)\left(a\beta_2'-3\alpha_2'\tau_1'\right)-\alpha_2'\left(\alpha_2'+a\beta_2\right)=0.$$

Analyzing the obtained ODE system, find the functions $f(y)$, for which the equations

$$\frac{d^2y}{dt^2}=\frac{dy}{dt}+f(y)$$

allows the dynamic symmetry of the required type.

Let us note here several parametrically given functions $f(y)$:

1. $$f(p)=-\frac{e^{-\frac{1}{2p+1}}(2p+3)^2}{8f_1\,(2p+1)},\quad y(p)=\frac{4e^{-\frac{1}{2p+1}}(p+1)}{f_1\,(2p+1)}+f_2;$$

2. $$f(p)=-\frac{e^{-\frac{1}{2p+1}}\left((3f_1+2f_1p)^2-(2p+1)^2\right)f_2}{8{f_1}^3\,(2p+1)},\quad y(p)=\frac{4e^{-\frac{1}{2p+1}}(p+1)\,f_2}{f_1\,(2p+1)}+f_4;$$

3. $$f(p)=-\frac{e^{-\frac{1}{2p+1}}\left((3f_1+2f_1p)^2+(2p+1)^2\right)f_2}{8{f_1}^3\,(2p+1)},\quad y(p)=\frac{4e^{-\frac{1}{2p+1}}(p+1)\,f_2}{f_1\,(2p+1)}+f_3;$$

4. $$f(p)=\frac{p^{k_1}\left(-k_1p^2+p+2pk_1+2f_1f_2\right)}{2f_2\,(p-1)},\quad y(p)=-\frac{(2k_1+1)^2\,(k_1-pk_1+1)\,p^{k_1}}{2f_2k_1\,(k_1+1)}+f_3;$$

5. $$f(p)=\frac{p^{k_1}\left(2p^2f_3+2pk_1+p-1-k_1\right)}{2f_2\,(p-1)},\quad y(p)=-\frac{(2k_1+1)^2\,(k_1-pk_1+1)\,p^{k_1}}{2f_2k_1\,(k_1+1)}+f_4;$$

6. $$f(p)=\frac{f_1(p+2)^2}{(p+1)},\quad y(p)=-4f_1\left(\ln\frac{p}{p+1}+\frac{1}{p+1}\right)+f_3;$$

7. $$f(p)=\frac{f_1(ph+2+p)(ph-2-p)}{(p+1)\,(h^2-1)},\quad y(p)=\frac{4f_1}{(h^2-1)}\left(\ln\frac{p}{p+1}+\frac{1}{p+1}\right)+f_3;$$

8. $$f(p)=\frac{f_1\left(p^2h^2+p^2+4p+4\right)}{(p+1)\,(h^2+1)},\quad y(p)=-\frac{4f_1}{(h^2-1)}\left(\ln\frac{p}{p+1}+\frac{1}{p+1}\right)+f_3;$$

9. $$f(p)=f_1\,(p+1)-f_2,\quad y(p)=f_2\left(\ln\frac{p}{p+1}+\frac{1}{p+1}\right)+f_3;$$

where p,g -parameters, f_1,f_2,f_3,f_4,k_1,h are arbitrary constants. It is interesting to note, that the variants 4., 5., for some values of arbitrary constants allow the explicit representation of the functions $f(y)$. Thus it is possible to obtain both known and new cases of the equations (7) integration.

The usage of the dynamic symmetry together with general solutions for the indicated nine cases are given in the following section.

3. EXAMPLE OF THE DYNAMIC SYMMETRIES USAGE FOR INTEGRATION OF SECOND ORDER ODE

Consider the procedure of dynamic symmetries usage for the equation (7) with the first type function $f(y)$. Taking into account that the function $\tau_1(y)$ here is representative in the parametric form

$$\tau_1(p)=-\frac{2}{2p+1},\quad y(p)=\frac{4e^{-\frac{1}{2p+1}}(p+1)}{f_1\,(2p+1)}+f_2,$$

pass in the equation (8) from the function $y(x)$ to the function $p(x)$. After transformation the following equation is obtained

$$\begin{gathered}
32a(2p+1)^2p''+64{p'}^3+48a(2p+1)^2p{p'}^2+\\
+4a^2\left(12p^2+36p+19\right)(2p+1)^2p'+\\
+a^3(2p+3)^2(2p+1)^4=0.
\end{gathered}\tag{9}$$

Invariant u takes the form

$$u=\frac{2a{f_1}^2(2p+1)^2-2a^2p+8{f_1}^2p'-a^2}{a\,(2p+1)}.\tag{10}$$

Using this invariant as new variable, low the order of the equation (9), obtaining the equation

$$
\begin{gathered}
32{f_1}^4u' + au^3 + a\left(8{f_1}^2 + 3a\right)u^2 + \\
+a\left(a + 4{f_1}^2\right)\left(3a + 4{f_1}^2\right)u + a^2\left(a + 4{f_1}^2\right)^2 = 0.
\end{gathered} \tag{11}
$$

Integrating the equation (11), it is not difficult to write out the general solution of the equation (9)

$$
\begin{gathered}
p = \frac{e^{-\frac{2}{g}}}{2\int \frac{e^{-\frac{2}{g}}(g-1)}{g^2}dg + C_2} - \frac{1}{2}, \\
x = \frac{2 - 2g + 2g\ln g}{ag} + C_1.
\end{gathered} \tag{12}
$$

Taking into account the connection between variables (p, x) and (y, t)

$$
y = \frac{4e^{-\frac{1}{2p+1}}(p+1)}{f_1(2p+1)} + f_2, \quad t = ax - \frac{2}{2p+1},
$$

it is possible to say about general solution of the equation

$$
\frac{d^2y}{dt^2} = \frac{dy}{dt} + f(y)
$$

with the function $f(y)$ of the first type.

Give here general solutions with functions $f(y)$ of other types:

2. $$y = \frac{4e^{-\frac{1}{2p+1}}(p+1)f_2}{f_1(2p+1)} + f_4, \quad t = x - \frac{2}{2p+1},$$

   where

   $$p = -\frac{1}{2} + \frac{g^{f_1}}{C_2 - \int \frac{g^{f_1}(g + f_1 g + 1 - f_1)}{g(g-1)}dg},$$

   $$x = -f_1\ln g - \ln\frac{(g-1)^2}{g} + C_1.$$

3. $$y = \frac{4e^{-\frac{1}{2p+1}}(p+1)f_2}{f_1(2p+1)} + f_3, \quad t = x - \frac{2}{2p+1},$$

   where

   $$p = -\frac{1}{2} + \frac{e^{-2f_1\arctan g}}{C_2 + 2({f_1}^2 + 1)\int \frac{e^{-2f_1\arctan g}}{(f_1 - g)(1 + g^2)}dg},$$

   $$x = \ln\frac{(1+g^2)}{(g-f_1)^2} + 2f_1\arctan g + C_1.$$

4. $$y = -\frac{(2k_1+1)^2(k_1 - pk_1 + 1)p^{k_1}}{2f_2k_1(k_1+1)} + f_3,$$
   $$t = x + (2k_1+1)\ln p,$$

   by denoting $m = 2k_1 + 1$ this solution takes the form

   $$p = \frac{e^{8{f_2}^2\int \frac{mg - 4{f_2}^2}{gB}dg}}{8m{f_2}^2\int \frac{e^{8{f_2}^2\int \frac{mg-4{f_2}^2}{gB}dg}}{B}dg + C_2},$$

   $$x = 32{f_2}^4 m\int \frac{dg}{gB} + C_1,$$

   where

   $B = g^2m^2(m + 1 + 4f_1f_2) - 8m{f_2}^2g - 16{f_2}^4(m-1).$

5. $$y = -\frac{(2k_1+1)^2(k_1 - pk_1 + 1)p^{k_1}}{2f_2k_1(k_1+1)} + f_4,$$
   $$t = x + (2k_1+1)\ln p,$$

   by denoting $m = 2k_1 + 1$ this solution takes the form

   $$p = -\left(8m{f_2}^2\int \frac{e^{-8{f_2}^2\int \frac{mg+4{f_2}^2}{gB}dg}}{B}dg + C_2\right)\times$$
   $$\times e^{8{f_2}^2\int \frac{mg+4{f_2}^2}{gB}dg},$$

   $$x = -32{f_2}^4 m\int \frac{dg}{gB} + C_1,$$

   where

   $B = g^2m^2(-m + 1 - 4f_3) + 8m{f_2}^2g + 16{f_2}^4(m+1).$

6. $$y = -4f_1\left(\ln\frac{p}{p+1} + \frac{1}{p+1}\right) + f_3, \quad t = x + \ln\frac{p}{p+1},$$

   where

   $$p = -1 + \frac{8{f_1}^2e^g}{(1-g)\left(8{f_1}^2\int \frac{ge^g}{(g-1)^2}dg + C_2\right)},$$

   $$x = C_1 - g - \ln(1-g).$$

7. $$y = \frac{4f_1}{(h^2-1)}\left(\ln\frac{p}{p+1} + \frac{1}{p+1}\right) + f_3, t = x + \ln\frac{p}{p+1},$$

   where

   $$p = -1 - \frac{16hg^{\frac{h+1}{2h}}{f_1}^2}{B\left(8{f_1}^2(h^2-1)\int \frac{g^{\frac{1-h}{2h}}(1+g)}{B^2}dg + C_2\right)},$$

   $$x = C_1 + \frac{(1-h^2)\int \frac{1+g}{gB}dg}{2h},$$

   $$B = gh - h - 1 - g.$$

8. $$y = -\frac{4f_1}{(h^2+1)}\left(\ln\frac{p}{p+1} + \frac{1}{p+1}\right) + f_3, t = x + \ln\frac{p}{p+1},$$

   where

   $$p = -1 + \frac{8{f_1}^2h\sqrt{1+g^2}e^{-\frac{\arctan g}{h}}}{(1-gh)\left(8(h^2+1){f_1}^2B + C_2\right)},$$

   $$x = C_1 + \frac{2\arctan g + h\ln(1+g^2) - 2h\ln(gh-1)}{2h},$$

   $$B = \int \frac{\sqrt{1+g^2}e^{-\frac{\arctan g}{h}}}{h^2g^4 - 2hg^3 + g^2 - 2hg + 1 + h^2g^2}dg.$$

9. $$y = f_2\left(\ln\frac{p}{p+1} + \frac{1}{p+1}\right) + f_3, t = x + \ln\frac{p}{p+1},$$

   where

   $$p = -\frac{g^{\frac{f_1}{f_2}-1}e^{-\frac{1+f_1g}{f_2g}}}{\int \frac{(1+f_1g)g^{\frac{f_1}{f_2}}e^{-\frac{1+f_1g}{f_2g}}}{f_2g^3}dg - C_2},$$

   $$x = C_1 + \frac{1 + f_1g(1-\ln g)}{f_2g}.$$

The possibilities of practical applications of indicated solutions are given in the following section.

4. SOLUTIONS OF TRAVELING WAVE TYPE FOR NONLINEAR HEAT CONDUCTION EQUATION

In the work [Berkovich (1992)] it is indicated that the equation (7) is connected with the Kolmogorov-Petrovsky-Piskunov diffusion equation and with the Semyonov equation (Fitz-Hugo-Nagumo) in the theory of the chain chemical reaction.

Consider the nonlinear heat conduction equations

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} - F(u) = 0. \tag{13}$$

If the function $F(u)$ satisfies the conductions

$$\begin{gathered} F(0) = F(1) = 0, \quad F'(0) = \alpha > 0, \\ F'(u) < \alpha, \quad 0 < u < 1, \end{gathered} \tag{14}$$

we can say about the Kolmogorov-Petrovsky-Piskunov equation.

If the function $F(u)$ satisfies the conductions

$$\begin{gathered} F(0) = F(1) = 0, \quad F(a_1) = 0, \quad 0 < a_1 < 1, \\ F'(0) \le 0, \quad F'(1) < 0, \quad F'(a_1) > 0, \end{gathered} \tag{15}$$

we can say about the Semyonov equation.

Let us find the traveling wave type solutions for the equations (13)

$$u(x,\tau) = y(\rho), \qquad \rho = x + b\tau.$$

transferring to the equation

$$\frac{dy}{d\rho} b - \frac{d^2 y}{d\rho^2} - F(y) = 0. \tag{16}$$

After the change of variables $t = b\rho$, one has

$$\frac{d^2 y}{dt^2} = \frac{dy}{dt} - \frac{F(y)}{b^2}.$$

Thus we obtain the equation of the type (7) under the condition $f(y) = -F(y)\big/b^2$. Let us show now, that the fifth type of the function $f(y)$ allows to write out the exact solutions of Kolmogorov-Petrovsky-Piskunov and Semyonov equations.

Setting the values of the constants $k_1 = 1$, $f_2 = 96$, $f_3 = 0.3$, $f_4 = 0.002$, obtain

$$f(p) = \frac{p(3p^2 + 15p - 10)}{960(p-1)}, \quad y(p) = -\frac{3}{64}p + \frac{3}{128}p^2 + \frac{1}{500}.$$

The relevant type of the functions $F(y)$ up to the expansion on the axis F has the form (Fig. 1).

Setting the values of the constants $k_1 = 1$, $f_2 = 10$, $f_3 = 1.97$, $f_4 = 0.149$, obtain

$$f(p) = \frac{p(197p^2 + 150p - 100)}{1000(p-1)}, \quad y(p) = -\frac{9}{20}p + \frac{9}{40}p^2 + \frac{149}{1000}.$$

The relevant type of the functions $F(y)$ up to the expansion on the axis F has the form (Fig. 2).

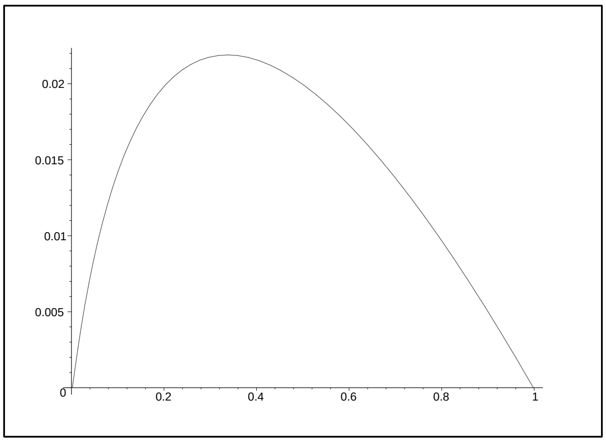

Fig. 1. Example of function $F(u)$ for Kolmogorov-Petrovsky-Piskunov equation

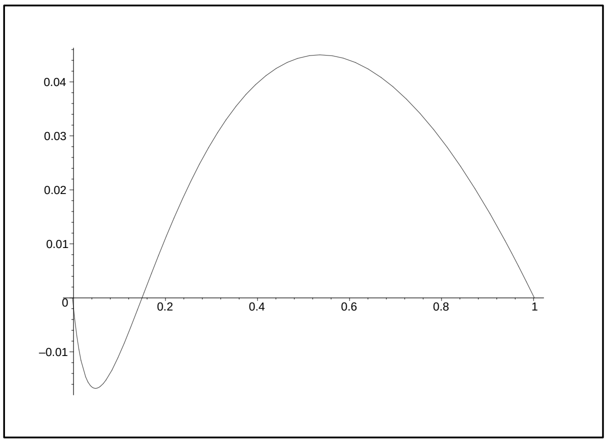

Fig. 2. Example of function $F(u)$ for Semyonov equation

Note that the fifth type of the function $f(y)$ representation by means of the constants change allows describing a wide class of solutions for Kolmogorov-Petrovsky-Piskunov and Semyonov equations.

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