$\lambda-$Symmetries and linearization of ordinary differential equations through nonlocal transformations ⋆

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Abstract: A recent study (Muriel and Romero (2010a)) on the linearization of ordinary differential equations through generalized Sundman transformations suggests considering the problem of linearization through nonlocal transformations from the point of view of the $\lambda-$symmetries admitted by the equation and their associated first integrals. The systematic methods to calculate $\lambda-$symmetries and associated first integrals of the ordinary differential equations provide well-defined algorithms to calculate the linearizing nonlocal transformations. Several examples, including differential equations with trivial algebra of Lie point symmetries, illustrate the presented methods.

Keywords: $\lambda-$Symmetries, linearization, generalized Sundman transformations, first integrals

1. INTRODUCTION

Certain classes of nonlinear ordinary differential equations (ODEs) can be solved if there exist local changes of variables that transform the ODEs into linear ODEs. This is a classical problem in the theory of differential equations firstly studied by S. Lie in 1883. Second-order ODEs that can be transformed into the linear *free particle* equation

$$X_{TT} = 0 \tag{1}$$

through local transformations of the form

$$X = R(t, x), T = S(t, x), \tag{2}$$

must be of the form

$$\ddot{x} + A_3(t, x)\dot{x}^3 + A_2(t, x)\dot{x}^2 + A_1(t, x)\dot{x} + A_0(t, x) = 0 \tag{3}$$

where over dot denotes differentiation of the dependent variable x with respect to the independent variable t. The functions A_0, A_1, A_2 and A_3 should satisfy two conditions involving their partial derivatives (Ibragimov (2006), Meleshko (2006)). This characterization is known as the Lie linearization test. The determination of functions R and S can be done by solving an involutive system of partial differential equations (PDEs). The two independent integrals of motion of the free particle equation

$$I_1 = X_T \text{ and } I_2 = X - TX_T, \tag{4}$$

can be used to derive, by inverting (2), two independent first integrals of the associated nonlinear equation and hence its general solution.

Since there are many ODEs that do not pass the Lie test of linearization, there have been many recent studies that consider transformations involving nonlocal terms to linearize a wider class of nonlinear ODEs (Duarte et al. (1994), Chandrasekar et al. (2006a), Chandrasekar et al. (2006b), Euler and Euler (2004)). The most simple type of such transformations have been named generalized Sundman transformations (Euler et al. (2003)) and are of the form

$$X = F(t, x), dT = G(t, x)dt. \tag{5}$$

In this paper the term $S-$*linearizable* refers to the ODEs that can be transformed into the linear equation (1) through a generalized Sundman transformation (5). Such ODEs must be of the form

$$\ddot{x} + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0 \tag{6}$$

and the coefficients a_0, a_1, a_2 should satisfy one of two pairs of two differential equations (see Duarte et al. (1994) or equations (29)-(30) below). These conditions give a criterion to check whether an ODE of the form (6) is $S-$linealizable or not. However, once it is known that a given ODE is $S-$linealizable, it remains, as in the case of local transformations, the problem of determining functions F and G that define the nonlocal linearizing transformation (5). A disadvantage over the linearization through local transformations is that the general solution of the nonlinear ODE can not be obtained straightforwardly by the general solution of the free particle equation, due to the nonlocal nature of (5). Nevertheless, the first integral $I_1 = X_T$ of (1) provides, by using (5), the following first integral of the nonlinear ODE

$$I = \frac{F_x(t, x)\dot{x} + F_t(t, x)}{G(t, x)}. \tag{7}$$

This fact allows us to connect the class of the $S-$linearizable equations with the class of equations that admit first integrals that are linear in the first order derivative (Muriel and Romero (2009b)). First integrals of ODEs are correlated with the $\lambda-$symmetries admitted by the ODE (Muriel and Romero (2009a), Muriel and Romero

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(2010b)). Since $S-$linearizable ODEs admit first integrals of the form (7), they should admit $\lambda-$symmetries of an specific form that can be used to give a novel characterization of these equations. Such $\lambda-$symmetries and the associated first integrals can be calculated by systematic methods, which allow us to overcome the difficulties in solving the involutive systems of PDEs to construct the nonlocal linearizing transformations (5).

There are ODEs of the form (6) that are not $S-$linearizable. These ODEs may still admit $\lambda-$symmetries with similar structure to the $\lambda-$symmetries of the $S-$linearizable equations. This allows us to extend the class of nonlinear ODEs that can be linearized but now considering more general types of nonlocal transformations:

$$X = F(t,x), dT = G(t,x,\dot{x})dt. \tag{8}$$

Some special classes of nonlocal transformations of type (8) have been considered in the literature. For instance, in Chandrasekar et al. (2006b), the authors have considered polynomials of first degree in $\dot{x}$ for $G(t,x,\dot{x})$. As far as we know, no feasible algorithm to obtain the linearizing nonlocal transformations (8) has been given. In this paper such transformations arise from the $\lambda-$symmetries and the associated first integrals of the nonlinear ODEs.

The paper is organized as follows: in Section 2 we explain, from a theoretical point of view, the general procedure to derive linearizing nonlocal transformations. In Section 3 these techniques are applied to a family of ODEs that includes relevant equations of the mathematical physics as particular cases. The case of ODEs linearizable by generalized Sundman transformations is discussed in Section 4 as a particular case of the general procedure. Significant simplifications of the general case can be performed for $S-$linearizable equations, which provide an effective tool to linearize wide classes of nonlinear ODEs. This is illustrated in Section 5 with an example of a second-order ODE that lacks Lie point symmetries and therefore does not pass the Lie test of linearization.

2. THE GENERAL PROCEDURE

Although for ODEs of order higher than one there are first integrals or integrating factors not associated to Lie point symmetries, it can be shown that there is an equivalence between first integrals and $\lambda-$symmetries [1]. The canonical pair of the equivalence class of the $\lambda-$symmetries associated to a first integral I is given by $(v,\lambda) = (\partial_x, -I_x/I_{\dot{x}})$ (Muriel and Romero (2009a), Muriel and Romero (2010b)). In particular, the first integrals of the form (7) are associated to the $\lambda-$symmetries equivalent to the canonical pair

$$(v,\lambda) = (\partial_x, \alpha(t,x)\dot{x} + \beta(t,x)) \tag{9}$$

where

$$\alpha = -\left(\frac{F_x}{G}\right)_x \left(\frac{F_x}{G}\right)^{-1}, \quad \beta = -\left(\frac{F_t}{G}\right)_x \left(\frac{F_x}{G}\right)^{-1} \tag{10}$$

This motivates us to consider the second-order ODEs of the form (6) that admit $\lambda-$symmetries of the form (9). In this case functions α and β should satisfy the following system of determining equations:

$$\alpha_x + \alpha^2 + a_2\alpha + a_{2x} = 0, \tag{11}$$

$$\beta_x + 2(a_2+\alpha)\beta + a_{1x} + \alpha_t = 0, \tag{12}$$

$$\beta_t + \beta^2 + a_1\beta + a_{0x} - a_0\alpha = 0. \tag{13}$$

The analysis of this system of PDEs can be carried out in a systematic way: equation (11) is a Riccati-type equation with a known particular solution $\alpha = -a_2$ and therefore its general solution, involving an arbitrary function $H_1(t)$, can readily be obtained. After substitution, equation (12) becomes a linear first order ODE, in which t is considered as a parameter. Its general solution depends on a function $H_2(t)$. Finally, equation (13) is used to set appropriated functions H_1 and H_2 in order to get solutions for α and β.

At this point we would like to remark that even though the Lie algorithm is in principle a straightforward one, the group classification of the ODEs involving arbitrary functions, such as equation (6), requires the integration of complicated overdetermined systems of PDEs for the infinitesimals of a Lie point symmetry. Usually special forms of the arbitrary functions must be chosen in order to make some progress in the symmetry analysis. This has not been necessary for the analysis of system (11)-(13). In fact, there are many ODEs for which system (11)-(13) is compatible but not the determining systems for the Lie point symmetries (see, for instance, equation (40) below).

Once particular solutions α and β of system (11)-(13) have been calculated (provided they exist), we construct a first integral I of the ODE associated to the $\lambda-$symmetry (9). This can be done by following the method proposed in Muriel and Romero (2009a):

(1) Find a first integral $w(t,x,\dot{x})$ of $v^{[\lambda,(1)]}$, i.e., a particular solution of the equation

$$w_x + (\alpha\dot{x} + \beta)\cdot w_{\dot{x}} = 0. \tag{14}$$

Such particular solution can be given by $w = A(t,x)\dot{x} + B(t,x)$, where A and B are any functions such that

$$-A_x/A = \alpha \text{ and } -B_x/A = \beta. \tag{15}$$

(2) Evaluate $Z(w)$ and express $Z(w)$ in terms of (t,w) as $Z(w) = \phi(t,w)$, where Z denotes the vector field associated to (6).

(3) Find a first integral $J = J(t,w)$ of $\widetilde{Z} = \partial_t + \phi(t,w)\partial_w$.

(4) Evaluate $I(t,x,\dot{x}) = J(t, w(t,x,\dot{x}))$.

Different families of nonlocal transformations (8) can be constructed by imposing $X_T = I$. Such transformations are defined by any pair of functions $F(t,x)$ and $G(t,x,\dot{x})$ such that

$$F_x(t,x)\dot{x} + F_t(t,x) = I(t,x,\dot{x})G(t,x,\dot{x}). \tag{16}$$

3. AN APPLICATION: NONLINEAR OSCILLATORS

Let us consider the family of equations

$$\ddot{x} + a_1(x)\dot{x} + a_0(x) = 0, \tag{17}$$

where

$$\begin{aligned} a_1(x) &= x\,f'(x) + 2f(x) + c_1, \\ a_2(x) &= (f^2(x) + c_1 f(x) + c_2)x. \end{aligned} \tag{18}$$

[1] $\lambda-$Symmetries were first introduced in Muriel and Romero (2001) to extend classical symmetry methods to integrate or reduce the order of ODEs. See Gaeta (2009) and references therein for a complete overview of this theory and some of its developments.

In (18) $f(x)$ is an arbitrary differentiable function and c_1 and c_2 are arbitrary constants. Several well-known equations representing physically important oscillator systems are particular cases of (17):

- For $f(x) = k\,x$ and $c_1 = 0$ equation (17) is the Modified Emden type equation with additional linear forcing:
$$\ddot{x} + 3k\,x\,\dot{x} + k^2x^3 + c_2x = 0. \tag{19}$$
- For $f(x) = k\,x$ we obtain the generalized modified Emden type equation (MEE):
$$\ddot{x} + (3k\,x + c_1)\,\dot{x} + k^2x^3 + c_1\,kx^2 + c_2\,x = 0. \tag{20}$$
- For $f(x) = k\,x^2$ equation (17) becomes the generalized force-free Duffing-van der Pol oscillator (DVP):
$$\ddot{x} + (4k\,x^2 + c_1)\,\dot{x} + k^2x^5 + k\,c_1\,x^3 + c_2\,x = 0. \tag{21}$$

Different choices of $f(x), c_1$ and c_2 generate a wide class of nonlinear ODEs.

The general solution of corresponding equation (11) is $\alpha(t,x) = 1/(x + H_1(t))$, where $H_1(t)$ is an arbitrary function. For $H_1(t) = 0$, equation (12) becomes
$$\beta_x + 2\beta/x + 3f' + xf'' = 0 \tag{22}$$
and $\beta = H_2(t)/x^2 - xf'(x)$ is its general solution, where $H_2(t)$ is an arbitrary function. It can be checked that for $H_2(t) = 0$, corresponding equation (13) is satisfied. Hence $\alpha = 1/x$ and $\beta = -xf'(x)$ solve the corresponding system (11)-(13).

(1) The particular solutions of (15), $A(t,x) = 1/x$ and $B(t,x) = f(x) + c_1/2$, can be easily calculated and hence
$$w = \dot{x}/x + f(x) + c_1/2 \tag{23}$$
is a particular solution of (14).

(2) We calculate $Z(w) = -(w^2 + k)$, where
$$k = c_2 - c_1^2/4 \tag{24}$$
and Z is the vector field associated to equation (17).

(3) First integrals of $\widetilde{Z} = \partial_t - (w^2+k)\partial_w$ arise as solutions of the Riccati-type ODE:
$$\rho'(t) + \rho(t)^2 + k = 0. \tag{25}$$
First integrals of $\widetilde{Z}$, depending on the values of k, are given by:
(a) $k = 0 : J(t,w) = t - 1/w$
(b) $k \neq 0 : J(t,w) = \frac{1}{\sqrt{k}}\arctan\left(\frac{w}{\sqrt{k}}\right) + t$

(4) We replace w by $\dot{x}/x + f(x) + c_1/2$ in each function $J(t,w)$ to obtain a first integral of (17) for each case.

A wide family of functions $F(t,x)$ and $G(t,x,\dot{x})$ that satisfies (16) can be derived. For example, the pair
$$F(t,x) = \varphi(t)$$
$$G(t,x,\dot{x}) = \frac{\varphi'(t)\sqrt{k}}{\sqrt{k}t + \arctan\left(\frac{\dot{x}/x + f(x) + c_1/2}{\sqrt{k}}\right)} \tag{26}$$
where φ is an arbitrary differentiable function, defines through (8) a family of nonlocal transformations that linearizes all the ODEs of the family (17) for which $c_1^2 \neq 4c_2$.

Similarly,
$$F(t,x) = \varphi(t)$$
$$G(t,x,\dot{x}) = \frac{\varphi'(t)}{t}\left(1 + \frac{x}{t(\dot{x} + f(x) + c_1) - x}\right) \tag{27}$$
defines through (8) a family of nonlocal transformations that linearizes all the ODEs of the family (17) for which $c_1^2 = 4c_2$.

4. THE PARTICULAR CASE OF $S-$LINEARIZABLE EQUATIONS

The general procedure described in Section 2 becomes surprisingly simple when $S-$linearizable ODEs are considered. Particular solutions of system (11)-(13) may be easily derived and adequate particular solutions of (15) can be chosen in order to get $Z(w) = 0$. In this case $J(t,w) = w$ is a first integral of $\partial_t + \phi(t,w)\partial_w$ and $I(t,x,\dot{x}) = w(t,x,\dot{x}) = A(t,x)\dot{x} + B(t,x)$ becomes a first integral of the ODE. By (16), any pair of functions $F = F(t,x)$ and $G = G(t,x)$ such that
$$F_x/G = A \text{ and } F_t/G = B \tag{28}$$
defines a generalized Sundman transformation (5) that linearizes the nonlinear ODE.

Next we describe a method to construct particular solutions (α,β) of system (11)-(13) and the appropriate solutions (A,B) of (15). The $S-$linearizable ODEs can be grouped in two different classes (Muriel and Romero (2010a)):

Type I: The coefficients of (6) satisfy
$$S_1 \equiv a_{1x} - 2a_{2t} = 0, \tag{29}$$
$$S_2 \equiv (a_0a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1 = 0.$$

Type II: The coefficients of (6) satisfy $S_1 \neq 0$ and
$$S_3 \equiv \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x}) = 0, \tag{30}$$
$$S_4 \equiv \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0x} = 0.$$

The $S-$linearizable equations are the only ODEs of the form (6) for which system (11)-(13) is compatible for $\alpha = -a_2$ (see Theorem 3 in Muriel and Romero (2009b)).

For the equations of type I system (11)-(13) admits infinitely many solutions $(\alpha = -a_2, \beta)$ and β is of form $-a_1/2 + h$, where h denotes a particular solution of the Riccati-type equation
$$h'(t) + h^2(t) + f(t) = 0, \tag{31}$$
for
$$f(t) = a_0a_2 + a_{0x} - a_{1t}/2 - a_1^2/4. \tag{32}$$
By choosing two different solutions h_1 and h_2 of equation (31), we get two nonequivalent $\lambda-$symmetries, (v,λ_i), defined by $v = \partial_x$ and $\lambda_i = -a_2\dot{x} - a_1/2 + h_i$, for $i = 1,2$. The associated first integrals are functionally independent (Theorem 2.3 in Muriel and Romero (2010b)). Such first integrals are the form $I_i = A_i\dot{x} + B_i$, where A_i and B_i can be constructed as follows (see Theorem 2 in (Muriel and Romero (2009b))):

- Let $P = P(t,x)$ be a function such that
$$P_t = a_1/2, \qquad P_x = a_2 \tag{33}$$
and g_1, g_2 two nonconstant functions such that
$$g_i'(t) = h_i(t)g(t) \text{ for } \quad i = 1,2. \tag{34}$$

We define

$$A_i(t,x) = g_i(t)e^{P(t,x)} \text{ for } i = 1,2. \quad (35)$$

- Functions B_1 and B_2 arise as particular solutions of the system

$$(B_i)_t = a_0\, A_i, \qquad (B_i)_x = (a_1/2 - h_i)\, A_i. \quad (36)$$

For the equations of type II, system (11)-(13) admits only one solution of the form $(\alpha = -a_2, \beta)$ and β is uniquely defined by $\beta = S_2/S_1$. There exists (up to multipliers) a unique first integral of the form $I = A\dot{x} + B$ associated to $(\partial_x, -a_2 + S_2/S_1)$ and functions A and B can be calculated following the next algorithm:

- Let $P = P(t,x)$ be a function such that

$$P_t = a_1 + S_2/S_1, \qquad P_x = a_2. \quad (37)$$

We define

$$A(t,x) = e^{P(t,x)}. \quad (38)$$

- Function B arises as any particular solution of the system

$$B_t = a_0\, A, \quad B_x = -\,(S_2/S_1)\, A. \quad (39)$$

Once A and B have been determined, we use (28) to find functions F and G that define, through (5), linearizing $S-$transformations [2].

In next section we present an example to illustrate the described methods.

5. AN EXAMPLE: A TYPE II EQUATION

Let us consider the second-order nonlinear ODE

$$\ddot{x} + \frac{x-1}{x}\dot{x}^2 - \frac{te^{-x}}{x}\dot{x} + e^{-x} = 0. \quad (40)$$

It can be checked, by using (29) and (30), that

$$S_1 = e^{-x}t(x+1)/x^2, \quad S_2 = e^{-2x}t^2(x+1)/x^3 \quad (41)$$

and $S_3 = S_4 = 0$. Hence $\alpha = (1-x)/x$ and $\beta = S_2/S_1 = e^{-x}t/x$ solve system (11)-(13) and define, by (9), a $\lambda-$symmetry of (40). However, it can be checked that equation (40) does not admit Lie point symmetries.

The function $P(t,x) = x - \ln(x)$ is a particular solution of corresponding system (37) and, by (38), we get $A(t,x) = e^x/x$. A particular solution of corresponding (39) is given by $B(t,x) = t/x$.

We can use (28) to find $S-$linearizing transformations: the general solution of the first order linear partial differential equation $BF_x - AF_t = 0$

$$tF_x - e^xF_t = 0 \quad (42)$$

is given by

$$F(t,x) = \varphi\left(\frac{t^2}{2} + e^x\right). \quad (43)$$

By (28), $G = F_x/A$ is given by

$$G(t,x) = x\varphi'\left(\frac{t^2}{2} + e^x\right). \quad (44)$$

For any arbitrary differentiable function φ, (43) and (44) define, by means of (5), a generalized Sundman transformation that linearizes equation (40). It must be observed that equation (40) does not pass the Lie test of linearization.

[2] An alternative construction of the linearizing $S-$transformations, that do not use the functions A and B that define the first integral, has been derived in Muriel and Romero (2010a) (Theorem 3).

6. CONCLUDING REMARKS

A novel approach to the problem of linearization of ODEs through nonlocal transformations has been presented. It is based on the $\lambda-$symmetries admitted by the nonlinear ODE and their associated first integrals. The algorithmic methods to calculate $\lambda-$symmetries and associated first integrals may overcome the difficulties of other approaches that require solving involutive systems of second-order PDEs to construct the nonlocal linearizing transformations.

The presented methods are applied to several examples of second-order nonlinear ODEs, including equations with trivial algebra of Lie point symmetries that therefore do not pass the Lie test of linearization. In particular we have obtained a unified procedure to linearize a wide class of ODEs that includes well-known equations representing physically important oscillator systems.

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