# Conservation laws for a subclass of self-adjoint third order equations ⋆

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**Abstract:** In this work we consider a class of third-order nonlinear partial differential equation containing two un-specified coefficient functions of the dependent variable which include various integrable and nonintegrable equations. We determine the subclasses of these equations which are self-adjoint. By using a general theorem on conservation laws proved by Nail Ibragimov we find conservation laws for some of these partial differential equations without classical Lagrangians.

Keywords: Adjoint equation to nonlinear equations; Lagrangian; Symmetry of adjoint equations; Conservation laws.

## 1. INTRODUCTION

The study of integrable equations has arisen lot of attention in the last years. Among the integrable equations the study of peaked and cusped soliton equations has been considered in many papers. Wadati et al. (1979); Wadati et al (1980); Shimizu and Wadati (1980), proposed the cusp soliton, which is a kind of peaked soliton. Recently Quiao and Liu (2009) proposed a new completely integrable equation

$$u_t = \frac{1}{n}\left(\frac{1}{u^n}\right)_{xxx} - \frac{1}{n}\left(\frac{1}{u^n}\right)_x \tag{1}$$

with $n = 2$, which has no smooth solitons. Quiao and Liu (2009) proved that the new equation proposed Eq. (1) with $n = 2$ is completely integrable. It was shown that (2) has bi-Hamiltonian structure, and Lax pair that implies its integrability by the Inverse Scattering Transformation. By considering traveling-wave solutions the authors found one peak soliton solutions and three-peaks solitons solutions. The authors state that no smooth solitons were found for equation (1), although equation (1) is completely integrable. They claim to provide an integrable system with no smooth solitons.

In a recent work Gandarias and Bruzon (2010), we studied equation (1) from the point of view of the theory of symmetry reductions in partial differential equations. We have obtained the classical symmetries admitted by (1) for arbitrary $n$, then, we used the transformations groups to reduce the equations to ordinary differential equations. Physical interpretation of these reductions and some elementary solutions were also provided and we derived exact solutions for the integrable equation. Some of these solutions are smooth soliton solutions.

In Ibragimov (2007) (see also Ibragimov (2006)) a general theorem on conservation laws for arbitrary differential equation which does not require the existence of Lagrangians has been proved. This new theorem is based on the concept of adjoint equations for non-linear equations. The notion of self-adjoint equations has been also extended in Ibragimov (2007) to non-linear equations.

The aim of this paper is to determine, for the generalized class of equation

$$u_t = (g(u))_{xxx} - (f(u))_x\,. \tag{2}$$

the subclasses of equations which are self-adjoint. For these classes of self-adjoint equations we apply Lie classical method and determine the functions for which Eqs. (2) have additional symmetries. We also determine, by using the notation and techniques of the work Ibragimov (2007), some nontrivial conservation laws for Eqs. (2).

## 2. PRELIMINARIES

### *2.1 Adjoint and self-adjoint nonlinear equations*

The following definitions of adjoint equations and self-adjoint equations are applicable to any system of linear and non-linear differential equations, where the number of equations is equal to the number of dependent variables (see Ibragimov (2007)), and contain the usual definitions for linear equations as a particular case. Since we will deal in our paper with scalar equations, we will formulate these definitions in the case of one dependent variable only.

Consider an sth-order partial differential equation

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0 \tag{3}$$

with independent variables $x = (x^1, \ldots, x^n)$ and a dependent variable $u$, where $u_{(1)} = \{u_i\}$, $u_{(2)} = \{u_{ij}\}, \ldots$ denote the sets of the partial derivatives of the first, second, etc. orders, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. The adjoint equation to (3) is

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{4}$$

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with

$$F^*(x,u,v,u_{(1)},v_{(1)},\ldots,u_{(s)},v_{(s)}) = \frac{\delta(v\,F)}{\delta u}\,, \qquad (5)$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}} \qquad (6)$$

denotes the variational derivatives (the Euler-Lagrange operator), and $v$ is a new dependent variable. Here

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots$$

are the total differentiations.

Eq. (3) is said to be *self-adjoint* if the equation obtained from the adjoint equation (4) by the substitution $v = u$ :

$$F^*(x,u,u,u_{(1)},u_{(1)},\ldots,u_{(s)},u_{(s)}) = 0,$$

is identical with the original equation (3). In other words, if

$$\begin{aligned} &F^*(x,u,u_{(1)},u_{(1)},\ldots,u_{(s)},u_{(s)}) = \\ &\phi(x,u,u_{(1)},\ldots)\,F(x,u,u_{(1)},\ldots,u_{(s)}). \end{aligned} \qquad (7)$$

### *2.2 General theorem on conservation laws*

We use the following theorem on conservation laws proved in Ibragimov (2007).

*Theorem 1.* Any Lie point, Lie-Bäcklund or non-local symmetry

$$X = \xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i} + \eta(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u} \qquad (8)$$

of Eqs. (3) provides a conservation law $D_i(C^i) = 0$ for the simultaneous system (3), (4). The conserved vector is given by

$$\begin{aligned} C^i = \xi^i\mathcal{L} &+ W\left[\frac{\partial\mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) - \cdots\right] \\ &+ D_j(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) + \cdots\right] \\ &+ D_jD_k(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}} - \cdots\right] + \cdots\,, \end{aligned} \qquad (9)$$

where $W$ and $\mathcal{L}$ are defined as follows:

$$W = \eta - \xi^j u_j, \quad \mathcal{L} = v\,F\left(x,u,u_{(1)},\ldots,u_{(s)}\right). \qquad (10)$$

The proof is based on the following operator identity:

$$X + D_i(\xi^i) = W\frac{\delta}{\delta u} + D_i\mathcal{N}^i, \qquad (11)$$

where $X$ is the operator (8) taken in the prolonged form:

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \zeta_i\frac{\partial}{\partial u_i} + \zeta_{i_1i_2}\frac{\partial}{\partial u_{i_1i_2}} + \cdots,$$

$$\zeta_i = D_i(\eta) - u_jD_i(\xi^j), \quad \zeta_{i_1i_2} = D_{i_2}(\zeta_{i_1}) - u_{ji_1}D_{i_2}(\xi^j), \ \ldots\,.$$

Since we will deal with fourth-order equations, we will use Eqs. (9) in the following form:

$$\begin{aligned} C^i = \xi^i\mathcal{L} &+ W\left[\frac{\partial\mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right)\right] \\ &+ W\left[-D_jD_kD_l\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right)\right] \\ &+ D_j(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) + D_kD_l\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right)\right] \\ &+ D_jD_k(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}} - D_l\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right)\right] \\ &+ D_jD_kD_l(W)\left(\frac{\partial\mathcal{L}}{\partial u_{ijkl}}\right). \end{aligned} \qquad (12)$$

### *2.3 The class of self-adjoint equations*

Let us single out self-adjoint equations from the equations of the form (2),

$$u_t = (g(u))_{xxx} - (f(u))_x\,.$$

The result is given by the following statement.

*Theorem 2.* Eq. (2) is self-adjoint if and only it has the form

$$u_t - (au+b)_{xxx} + (f(u))_x = 0, \quad f(u) = \text{arbitrary}. \qquad (13)$$

**Proof.** Eq. (5) yields

$$\begin{aligned} &F^* = \frac{\delta}{\delta u}[v(u_t + (g(u))_{xxx} + (f(u))_x = 0] \\ &= g_u\,v_{xxx} + 3\,g_{uu}\,u_x\,v_{xx} + 6\,g_{uu}\,u_{xx}\,v_x \\ &+ 6\,g_{uuu}\,(u_x)^2\,v_x - f_u\,v_x - v_t + 3\,g_{uu}\,u_{xxx}\,v \\ &+ 9\,g_{uuu}\,u_x\,u_{xx}\,v + 3\,g_{uuuu}\,(u_x)^3\,v \end{aligned} \qquad (14)$$

Setting $v = u$ in (14) we have

$$\begin{aligned} F^* =& (3\,g_{uu}\,u + g_u)\,u_{xxx} + (9\,g_{uuu}\,u + 9\,g_{uu})\,u_x\,u_{xx} \\ &+ (3\,g_{uuuu}\,u + 6\,g_{uuu})\,(u_x)^3 - f_u\,u_x - u_t \end{aligned}$$

using Eq. (7) yields:

$$\begin{aligned} &(3\,g_{uu}\,u + g_u\,k + g_u)\,u_{xxx} \\ &+ (9\,g_{uuu}\,u + 3\,g_{uu}\,k + 9\,g_{uu})\,u_x\,u_{xx} \\ &+ (3\,g_{uuuu}\,u + g_{uuu}\,k + 6\,g_{uuu})\,(u_x)^3 \\ &+ (-f_u\,k - f_u)\,u_x + (-k-1)\,u_t = 0. \end{aligned} \qquad (15)$$

Comparing the coefficients for $u_t$, we obtain $\phi = -1$ in (7). Therefore (15) shows that the latter equation is satisfied if and only if $g(u)$ solves the differential equation $g'' = 0$, whence

$$g(u) = au + b, \quad f(u) = \text{arbitrary}.$$

## 3. CLASSICAL SYMMETRIES

In this section we perform Lie symmetry analysis for equation (2) with $g(u) = au + b$ $a = 1$

$$u_t = u_{xxx} - (f(u))_x\,. \qquad (16)$$

Let us consider a one-parameter Lie group of infinitesimal transformations in $(x,t,u)$ given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + \mathcal{O}(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,t,u) + \mathcal{O}(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x,t,u) + \mathcal{O}(\varepsilon^2), \end{aligned} \qquad (17)$$

where $\varepsilon$ is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of the equation (1). This yields to the overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}. \qquad (18)$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi\frac{\partial u}{\partial x} + \tau\frac{\partial u}{\partial t} - \phi = 0. \qquad (19)$$

By solving this system we get that

$$\tau = \tau(t)\;\; \xi = \frac{\tau'}{3}x + \delta(t)\;\; \phi = \alpha(t)u + \beta(x,t), \qquad (20)$$

where $\alpha(t)$, $\beta(x,t)$, $\tau(t)$ and $f(u)$ must satisfy the following equations

$$\alpha_t u + \beta_x f_u + \beta_{xxx} + \beta_t = 0,$$
$$\tau_{tt}x - 3\alpha f_{uu}u - 2\tau_t f_u - 3\beta f_{uu} + 3\delta_t = 0. \qquad (21)$$

Solving this system we find that:

If $f(u)$ is arbitrary, *the symmetries that are admitted* by (1) are

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}.$$

The only functionals forms of f(u) with $f(u) \neq constant$ and f(u) nonlinear which have extra symmetries are the following

**Table 1:** Symmetries for the generalized equation.

| $i$ | $f(u)$ | $V_3^i$ |
|---|---|---|
| 1 | $\frac{au^{(n+1)}}{n+1} + bu + c$ | $(\frac{nx}{3} + \frac{2bnt}{3})\frac{\partial}{\partial x} + nt\frac{\partial}{\partial t} - \frac{2}{3}u\frac{\partial}{\partial u}$ |
| 2 | $-\frac{d}{a}e^{-au} + c$ | $\frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2}{3a}\frac{\partial}{\partial u}$ |
| 3 | $log(u) + b$ | $x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}$ |

## 4. CONSERVATION LAWS

### *4.1 Conservation laws for a subclass of self-adjoint third order equations*

1 Let us apply Theorem 1 to the self-adjoint equation:

$$u_t + u_{xxx} + au^n u_x = 0. \qquad (22)$$

In this case we have

$$\mathcal{L} = \Big(u_t + u_{xxx} + au^n u_x = 0\Big)v. \qquad (23)$$

We will write generators of point transformation group admitted by Eq. (22) in the form

$$X = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}$$

by setting $t = x^1,\ x = x^2$. The conservation law will be written

$$D_t(C^1) + D_x(C^2) = 0. \qquad (24)$$

Since we deal with a third-order equation, we compute the conserved vectors by (12).

Let us find the conservation law provided by the following obvious scaling symmetry of Eq. (22):

$$X = nt\frac{\partial}{\partial t} + \frac{nx}{3}\frac{\partial}{\partial x} - \frac{2u}{3}\frac{\partial}{\partial u}. \qquad (25)$$

In this case we have $W = -\frac{2u}{3} - ntu_t - \frac{nx}{3}u_x$ and Eqs. (12) yield the conservation law (24) with

$$\begin{aligned}
C^1 &= nt\mathcal{L} + Wv = ntv(u_t + u_{xxxx} + au^n u_x) \\
&\quad -v(-\frac{2u}{3} - ntu_t - \frac{nx}{3}u_x) \\
&= -\frac{n\,u_x\,v\,x}{3} + n\,t\,u_{x\,x\,x}\,v + n\,t\,u^n\,u_x\,v - \frac{2\,u\,v}{3}
\end{aligned}$$

$$\begin{aligned}
C^2 &= \frac{nx}{3}\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial u_x} + D_{xx}\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right] \\
&\quad -D_x(W)\left[D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right)\right] + D_x^2(W)\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right) \\
&= -\frac{n\,u_x\,v_{x\,x}\,x}{3} + \frac{n\,u_{x\,x}\,v_x\,x}{3} + \frac{n\,u_t\,v\,x}{3} - n\,t\,u_t\,v_{x\,x} \\
&\quad -\frac{2\,u\,v_{x\,x}}{3} + \frac{n\,u_x\,v_x}{3} + \frac{2\,u_x\,v_x}{3} + n\,t\,u_{t\,x}\,v_x - \frac{2\,n\,u_{x\,x}\,v}{3} \\
&\quad -\frac{2\,u_{x\,x}\,v}{3} - n\,t\,u_{t\,x\,x}\,v - n\,t\,u^n\,u_t\,v - \frac{2\,u^{n+1}\,v}{3}
\end{aligned}$$

Setting $v = u$ we obtain

$$\begin{aligned}
C^1 &= -\frac{n\,u\,u_x\,x}{3} + n\,t\,u\,u_{x\,x\,x} + n\,t\,u^{n+1}\,u_x - \frac{2\,u^2}{3} \\
&= D_x\left(-\frac{n\,u^2\,x}{6} + n\,t\,u\,u_{x\,x} - \frac{n\,t\,(u_x)^2}{2} + \frac{n\,t\,u^{n+2}}{n+2}\right) \\
&\quad +\frac{(n-4)u^2}{6},
\end{aligned}$$

$$\begin{aligned}
C^2 &= \frac{n\,u\,u_t\,x}{3} - n\,t\,u_t\,u_{x\,x} - \frac{2\,n\,u\,u_{x\,x}}{3} - \frac{4\,u\,u_{x\,x}}{3} \\
&\quad +\frac{n\,(u_x)^2}{3} + \frac{2\,(u_x)^2}{3} + n\,t\,u_{t\,x}\,u_x \\
&\quad -n\,t\,u\,u_{t\,x\,x} - n\,t\,u^{n+1}\,u_t - \frac{2\,u^{n+2}}{3} \\
&= D_t\left(\frac{n\,u^2\,x}{6} - n\,t\,u\,u_{x\,x} + \frac{n\,t\,(u_x)^2}{2} - \frac{n\,t\,u^{n+2}}{n+2}\right) \\
&\quad +\frac{(n-4)}{6\,(n+2)}\Big(2\,n\,u\,u_{x\,x} + 4\,u\,u_{x\,x} - n\,(u_x)^2 \\
&\quad -2\,(u_x)^2 + 2\,u^{n+2}\Big).
\end{aligned}$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = \frac{(n-4)u^2}{6},$$

$$\begin{aligned}
C^2 = \frac{(n-4)}{6\,(n+2)}&\Big(2\,n\,u\,u_{x\,x} + 4\,u\,u_{x\,x} - n\,(u_x)^2 \\
&-2\,(u_x)^2 + 2\,u^{n+2}\Big).
\end{aligned}$$

Let us find the conservation law provided by the following obvious scaling symmetry of Eq.

$$u_t + u_{xxx} + de^{-au}u_x = 0. \tag{26}$$

$$X = t\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x} + \frac{2}{3a}\frac{\partial}{\partial u}. \tag{27}$$

In this case we have $W = \dfrac{2}{3a} - tu_t - \dfrac{x}{3}u_x$ and Eqs. (12) yield the conservation law (24) with

$$\begin{aligned} C^1 &= t\mathcal{L} + Wv = tv(u_t + u_{xxxx} + be^{-au}u_x) \\ &\quad -v(\frac{2u}{3} - tu_t - \frac{x}{3}u_x) \\ &= -\frac{u_x\, v\, x}{3} + t\, u_{x\,x\,x}\, v + f\, t\, u_x\, v - \frac{2\, k\, v}{3} \end{aligned}$$

$$\begin{aligned} C^2 &= \frac{x}{3}\mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_x} + D_{xx}\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right] \\ &\quad -D_x(W)\left[D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right)\right] \\ &\quad +D_x^2(W)\left(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right) \\ &= -\frac{u_x\, v_{x\,x}\, x}{3} + \frac{u_{x\,x}\, v_x\, x}{3} + \frac{u_t\, v\, x}{3} - t\, u_t\, v_{x\,x} \\ &\quad -\frac{2\, k\, v_{x\,x}}{3} + \frac{u_x\, v_x}{3} + t\, u_{t\,x}\, v_x - \frac{2\, u_{x\,x}\, v}{3} \\ &\quad -t\, u_{t\,x\,x}\, v - f\, t\, u_t\, v - \frac{2\, f\, k\, v}{3} \end{aligned}$$

Setting $v = u$ we obtain

$$\begin{aligned} C^1 &= -\frac{u\, u_x\, x}{3} + t\, u\, u_{x\,x\,x} + d\, t\, u\, e^{-a\,u}\, u_x + \frac{2\, u}{3\, a} \\ &= D_x\left(-\frac{u^2\, x}{6} + t\, u\, u_{x\,x} - \frac{t\, (u_x)^2}{2} \right. \\ &\quad \left. -\frac{d\, t\, u\, e^{-a\,u}}{a} - \frac{d\, t\, e^{-a\,u}}{a^2}\right) + \frac{u^2}{6} + \frac{2\, u}{3\, a}. \end{aligned}$$

$$\begin{aligned} C^2 &= \frac{u\, u_t\, x}{3} - t\, u_t\, u_{x\,x} - \frac{2\, u\, u_{x\,x}}{3} + \frac{2\, u_{x\,x}}{3\, a} + \frac{(u_x)^2}{3} \\ &\quad +t\, u_{t\,x}\, u_x - t\, u\, u_{t\,x\,x} - d\, t\, u\, e^{-a\,u}\, u_t + \frac{2\, d\, u\, e^{-a\,u}}{3\, a} \\ &= D_t\left(\frac{u^2\, x}{6} - t\, u\, u_{x\,x} + \frac{t\, (u_x)^2}{2} + \frac{d\, t\, u\, e^{-a\,u}}{a} \right. \\ &\quad \left. +\frac{d\, t\, e^{-a\,u}}{a^2}\right) + \frac{u\, u_{x\,x}}{3} + \frac{2\, u_{x\,x}}{3\, a} - \frac{(u_x)^2}{6} \\ &\quad -\frac{d\, u\, e^{-a\,u}}{3\, a} - \frac{d\, e^{-a\,u}}{a^2}. \end{aligned}$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = \frac{u^2}{6} + \frac{2\, u}{3\, a},$$

$$C^2 = \frac{u\, u_{x\,x}}{3} + \frac{2\, u_{x\,x}}{3\, a} - \frac{(u_x)^2}{6} - \frac{d\, u\, e^{-a\,u}}{3\, a} - \frac{d\, e^{-a\,u}}{a^2}.$$

3

Let us find the conservation law provided by the following obvious scaling symmetry of Eq.

$$u_t + u_{xxx} + \frac{u_x}{u} = 0. \tag{28}$$

$$X = t\frac{\partial}{\partial t} + \frac{x}{3}\frac{\partial}{\partial x} + \frac{2}{3}\frac{\partial}{\partial u}. \tag{29}$$

In this case we have $W = \dfrac{2u}{3} - tu_t - \dfrac{x}{3}u_x$ and Eqs. (12) yield the conservation law (24) with

$$\begin{aligned} C^1 &= -v(\frac{2u}{3} - tu_t - \frac{x}{3}u_x) \\ &= -\frac{u_x\, v\, x}{3} + t\, u_{x\,x\,x}\, v + \frac{t\, u_x\, v}{u} + \frac{2\, u\, v}{3} \end{aligned}$$

$$\begin{aligned} C^2 &= \frac{x}{3}\mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_x} + D_{xx}\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right] \\ &\quad -D_x(W)\left[D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right)\right] + D_x^2(W)\left(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right) \\ &= -\frac{u_x\, v_{x\,x}\, x}{3} + \frac{u_{x\,x}\, v_x\, x}{3} + \frac{u_t\, v\, x}{3} - t\, u_t\, v_{x\,x} \\ &\quad +\frac{2\, u\, v_{x\,x}}{3} - \frac{u_x\, v_x}{3} + t\, u_{t\,x}\, v_x - t\, u_{t\,x\,x}\, v \\ &\quad -\frac{t\, u_t\, v}{u} + \frac{2\, v}{3} \end{aligned}$$

Setting $v = u$ we obtain

$$\begin{aligned} C^1 &= -\frac{u\, u_x\, x}{3} + t\, u\, u_{x\,x\,x} + t\, u_x + \frac{2\, u^2}{3} \\ &= D_x\left(-\frac{u^2\, x}{6} + t\, u\, u_{x\,x} - \frac{t\, (u_x)^2}{2} + t\, u\right) + \frac{5\, u^2}{6}. \end{aligned}$$

$$\begin{aligned} C^2 &= \frac{u\, u_t\, x}{3} - t\, u_t\, u_{x\,x} + \frac{2\, u\, u_{x\,x}}{3} - \frac{(u_x)^2}{3} + t\, u_{t\,x}\, u_x \\ &\quad -t\, u\, u_{t\,x\,x} - t\, u_t + \frac{2\, u}{3} \\ &= D_t\left(\frac{u^2\, x}{6} - t\, u\, u_{x\,x} + \frac{t\, (u_x)^2}{2} - t\, u\right) \\ &\quad + \frac{5\, u\, u_{x\,x}}{3} - \frac{5\, (u_x)^2}{6} + \frac{5\, u}{3} \end{aligned}$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from $C^1$ to $C^2$ and obtain

$$C^1 = \frac{5\, u^2}{6}, \qquad C^2 = \frac{5\, u\, u_{x\,x}}{3} - \frac{5\, (u_x)^2}{6} + \frac{5\, u}{3}. \tag{30}$$

## 5. QUASI-SELF ADJOINTNESS AND BLUMAN AND ANCO METHOD

Many equations having remarkable symmetry properties and physical significance are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from conservation laws of these equations by setting v = u, Ibragimov (2007) generalized the concept of self-adjoint equations by introducing the definition of quasi-self-adjoint equations.

Equation (3) is said to be quasi-self-adjoint if the the adjoint equation (4) is equivalent to the original equation (3) upon the substitution $v = h(u)$ with a certain function $h(u)$ such that $h'(u) \neq 0$. We consider again (1) and we substitute

$$v = h(u) \quad v_t = h'u_t$$
$$v_x = h'u_x \quad v_{xx} = h'u_{xx} + h''u_x^2$$

in the adjoint equation

$$\frac{v_{xxx}}{u^3} + \frac{v_x}{u^3} - v_t = 0$$

and obtain

$$h'(\frac{u_x}{u^3} - \frac{u_{xxx}}{u^3} - u_t) - \frac{3h''u_xu_{xx}}{u^3} - \frac{h'''u_x^3}{u^3} = 0$$

hence

$$h'(\frac{u_x}{u^3} - \frac{u_{xxx}}{u^3} - u_t) - \frac{3h''u_xu_{xx}}{u^3} - \frac{h'''u_x^3}{u^3}$$
$$= \lambda\left[u_t - \frac{1}{2}\left(\frac{1}{u^2}\right)_{xxx} + \frac{1}{2}\left(\frac{1}{u^2}\right)_x\right].$$

Hence $\lambda = -h'$ and

$$uh'' + 3h' = 0,\ u^2h''' - 12h' = 0$$

that yields $h = \frac{k_1}{u^2}$. Namely the adjoint equation becomes equivalent to the original equation upon the substitution $v = \frac{1}{2u^2}$.

Anco and Bluman (2002) gives a general treatment of a direct conservation law method for partial differential equations expressed in a standard Cauchy-Kovaleskaya form

$$u_t = G(x, u, u_x, u_{xx}, \ldots, u_{nx}).$$

The nontrivial conservation laws are characterized by a multiplier $\lambda$ with no dependence on $u_t$ satisfying

$$\hat{E}[u]\left(\lambda u_t - \lambda G(x, u, u_x, u_{xx}, \ldots, u_{nx})\right) = 0.$$

Here

$$\hat{E}[u] := \frac{\partial}{\partial u} - D_t o\frac{\partial}{\partial u_t} - D_x o\frac{\partial}{\partial u_x} + -D_x^2 o\frac{\partial}{\partial u_{xx}} + \ldots.$$

The conserved current must satisfy

$$\lambda = \hat{E}[u]\Phi^t$$

and the flux $\Phi^x$ is given by

$$\Phi^x = -D_x^{-1}(\lambda G) - \frac{\partial \Phi^t}{\partial u_x}G + GD_x\left(\frac{\partial \Phi^t}{\partial u_{xx}}\right) + \ldots.$$

For (1) we by applying this method we get $\lambda = \frac{1}{u}$ and

$$\Phi^t = -\frac{1}{u},\ \Phi^x = -\frac{u_{xx}}{u^5} + \frac{2u_x^2}{u^6} - \frac{1}{4u^4}$$

and the conservation law

$$D_t(\Phi^t) - D_x(\Phi^x) = 0.$$

In a future work we plan to develop conservation laws for the general equation (2) when it is quasi-self adjoint as well as using Anco and Bluman method.

## 6. ACKNOWLEDGMENTS

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