Similarity and Traveling Wave Solutions for a $K(m,n)$ Equation ⋆

M.S. Bruzón [*] **M.L. Gandarias** [*]

[*] *Departamento de Matemáticas, Universidad de Cádiz, PO.BOX 40, 11510 Puerto Real, Cádiz, Spain (e-mail: matematicas.casem@uca.es).*

Abstract: We analyze the $K(m,n)$ equation by means of the theory of symmetry reductions of partial differential equations. It is found that some similarity solutions are solutions with physical interest: source solutions and traveling-wave solutions. Moreover some particular solutions among the above types of solutions are obtained.

Keywords:
Partial differential equations; Nonlinear; Symmetries; Exact solutions.

1. INTRODUCTION

Rosenau and Hyman Rosenau and Hyman [1993] studied the role of nonlinear dispersive in the formation of patterns in liquid drops of the nonlinear dispersive equations

$$u_t + u^m u_x + (u^n)_{xxx} = 0, \tag{1}$$

for $m > 0, 1 < n \leq 3$. They also introduced a class of solitary wave solutions with compact support, i.e. the absence of infinite wings or the absence of infinite tails, called *compactons*. In addition to compactons, Rosenau [1997] proved that the nonlinear dispersive equations $K(m,n)$

$$u_t \pm a(u^m)_x + (u^n)_{xxx} = 0, \qquad a \quad \text{const.},$$

which exhibit a number of remarkable dispersive effects, can support kinks and solitons with infinite slopes, periodic waves and dark solitons with cusps all being manifestations of nonlinear dispersion in action. For $n < 0$ the enhanced dispersion at the tail may generate algebraically decaying patterns. Other solitary-wave solutions of $K(m,n)$ equations were also found by Rosenau [1994, 1998].

There is no existing general theory for solving nonlinear partial differential equations (PDE's). Due to the great advance in computation in the last years a great progress is being made in the development of methods and their applications to nonlinear PDE's for finding exact solutions. For instance, classical Lie method, nonclassical method, classical potential symmetries method, nonclassical potential symmetries method, simplest equation method, (G'/G)-expansion method and extended simplest equation method were applied by Bruzón et al. [2007,2008], Bruzón and Gandarias [2008,2009], Bluman and Kumei [1989], Gandarias [1997], Kudryashov [2005], Bruzón [2009], Kudryashov and Loguinova [2008], among other.

Local symmetries admitted by a PDE are useful for finding invariant solutions. These solutions are obtained by using group invariants to reduce the number of independent variables. The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of the Lie group theory provides a systematic method to search for these special group invariant solutions. For PDE's with two independent variables, as equation (1) is, a single group reduction transform PDE's into ordinary differential equations (ODE's), which are generally easier to solve than the original PDE.

In this paper we study the $K(m,n)$ equation

$$u_t + a u^m u_x + b(u^n)_{xxx} = 0, \tag{2}$$

with $a, b, m, n \in \mathcal{R}^*$ from the point of view of the theory of symmetry reductions in partial differential equations. By using this theory, we find that for the $K(m,n)$ equation some similarity solutions are solutions with physical interest which is the case of traveling-wave solutions and source solutions.

2. CLASSICAL SYMMETRIES

To apply the classical method to equation (2) with $a, b \neq 0$ we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned}$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of (2). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{3}$$

Invariance of equation(2) under a Lie group of point transformations with infinitesimal generator (3) leads to a set of nineteen determining equations. The solutions of this system depend on the parameters of equation (2). If

⋆ The authors acknowledge the financial support from Junta de Andalucía group FQM–201, from project MTM2009-11875 and from project P06-FQM-01448.

a, b, n and m are arbitrary constants with $a, b, m, n \in R^*$ the symmetries admitted by (2) are the group of space and time translations, which are defined by the infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t,$$

and the generator

$$\mathbf{v}_3 = \frac{1}{2}(n-m-1)x\partial_x + \frac{1}{2}(n-3m-1)t\partial_t + u\partial_u.$$

If $m = n = 1$ we obtain a new symmetry for equation (2)

$$\mathbf{v}_4 = at\partial_x + \partial_u$$

In this case equation (2) is the Korteweg-de Vries equation. This equation has been studied extensively by Ablowitz and Segur [1981], Clarkson and Kruskal [1989], Das [1989], Miura [1976]. In particular, by means of the inverse scattering method, by applying the direct method that involves no group theoretical techniques and the Bäcklund transformation has been determined by Newell [1976].

Having determined the infinitesimals for the $k(m, n)$ equation, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi\partial_x + \tau\partial_t - \eta = 0. \tag{4}$$

REDUCTION 1. From $\mathbf{v}_1 + \mathbf{v}_2$ we obtain traveling wave reductions

$$z = \mu x - \lambda t, \qquad u = h(z),$$

where $h(z)$, after integrating once with respect to z, satisfies

$$\begin{aligned} & h'' + \frac{n-1}{h}\left(h'\right)^2 + \frac{a}{b\mu^2 n\,(m+1)} h^{m-n+2} - \frac{\lambda}{b\mu^3 n} h^{2-n} \\ & + \frac{k_1}{b\,\mu^3\,n}\, h^{1-n} = 0, \end{aligned} \qquad l(5)$$

where k_1 is an integrating constant.

REDUCTION 2. From $\mathbf{v}_3$ the similarity variable and the similarity solution are:

$$z = xt^{-\frac{n-m-1}{n-3\,m-1}}, \qquad u = t^{\frac{2}{n-3\,m-1}}\, h(z).$$

This transformation reduces equation (2) into

$$\begin{aligned} & bnh^{n+2}h''' - \frac{n-m-1}{n-3\,m-1} zh^3h' + \frac{2}{n-3\,m-1}h^4 \\ & + b\,(n-2)\,(n-1)\,nh^n\,\left(h'\right)^3 + 3b\,(n-1)\,nh^{n+1}h'h'' \\ & - a\,h^{m+3}h' = 0. \end{aligned} \tag{6}$$

ANALYSIS OF THE REDUCED EQUATIONS.

EQUATION (5).

By making the change of variables

$$h^n = y \tag{7}$$

equation (5) becomes

$$y'' = \frac{(m+1)\;y^{\frac{1}{n}}\,\lambda - a\,\mu\,y^{\frac{m+1}{n}} - k_1\,m - k_1}{(b\,m + b)\;\mu^3}. \tag{8}$$

After multiplying (8) by $2y'$ and integrating once with respect to z we get

$$\begin{aligned} (y')^2 = {} & \frac{2\,(m+1)\,y^{\frac{1}{n}+1}\,\lambda}{(b\,m\,\mu^3 + b\,\mu^3)\,\left(\frac{1}{n}+1\right)} \\ & - \frac{2\,a\,\mu\,y^{\frac{m}{n}+\frac{1}{n}+1}}{(b\,m\,\mu^3 + b\,\mu^3)\,\left(\frac{m}{n}+\frac{1}{n}+1\right)} \\ & - \frac{2\,k_1\,(m+1)\,y}{b\,m\,\mu^3 + b\,\mu^3} + k_2, \end{aligned} \tag{9}$$

where k_2 is an integrating constant.

From equation (9) we obtain the following solutions for equation (5).

Theorem 1. Equation (5) admits in the following cases solutions in terms of the Jacobi elliptic functions $h = \alpha\,\mathrm{sn}^\beta(z,k)$ or $h = \alpha\,\mathrm{cn}^\beta(z,k)$, where α and β are constants different to zero:

1. For $n = \frac{1}{2}$, $m = -\frac{1}{2}$,

1.1. $a = \frac{\sqrt{3}\,(k+1)\,\lambda}{3\,k\,\mu}\sqrt{-\frac{k}{\lambda}}$, $b = \frac{\sqrt{3}\,\lambda}{6\,k\,\mu^3}\sqrt{-\frac{k}{\lambda}}$, k is arbitrary, $\alpha = \dfrac{-3k}{\lambda}$ and $\beta = 4$.

1.2. $a = \frac{\sqrt{3}\,(2\,k-1)\,\lambda}{3\,k\,\mu}\sqrt{-\frac{k}{(k-1)\,\lambda}}$, $b = -\frac{\sqrt{3}\,\lambda}{6\,k\,\mu^3}\sqrt{-\frac{k}{(k-1)\,\lambda}}$, k is arbitrary, $\alpha = \dfrac{9\,k^2}{(k-1)^2\,\lambda^2}$ and $\beta = 4$.

1.3. $a = \frac{\sqrt{3}\,\sqrt{1-k}\,(2\,k-1)\,\sqrt{\lambda}}{3\,(k-1)\,\sqrt{k}\,\mu}$, $b = -\frac{\sqrt{3}\,\sqrt{1-k}\,\sqrt{\lambda}}{6\,(k-1)\,\sqrt{k}\,\mu^3}$, k is arbitrary, $\alpha = -\dfrac{3\,(k-1)}{k\,\lambda}$ and $\beta = -4$.

2. For $m = 2(n-1)$,

2.1. $a = \frac{\alpha^{2(1-n)}\,(2n-1)\,(3n-1)\,\lambda}{\mu\,(n+1)}$, $b = \frac{\alpha^{1-n}\,(1-n)^2\,\lambda}{2\,\mu^3\,n\,(n+1)}$, $k = -1$, α is arbitrary and $\beta = -\frac{2}{1-n}$.

2.2. $a = \frac{\alpha^{2(n-3)}\,(2n-1)\,(n+3)\,\lambda}{\mu\,(3(n-1))}$, $b = -\frac{\alpha^{-n+m-1}\,(3-n)^2\,\lambda}{2\,\mu^3\,n\,(3(n-1))}$, $k = -1$, α is arbitrary and $\beta = -\frac{2}{3-n}$.

3. For $n = 1$, $m = 2$, $a = \frac{6\,k\,\lambda}{\alpha^2\,(k+1)\,\mu}$, $b = -\frac{\lambda}{(k+1)\,\mu^3}$, k and α arbitraries and $\beta = 1$.

4. For $n = \frac{1}{3}$, $m = -\frac{2}{3}$, $a = \dfrac{\left(\alpha^{\frac{2}{3}}\,(k+1)\right)\lambda}{6\,k\,\mu}$, $b = \frac{\alpha^{\frac{2}{3}}\,\lambda}{2\,k\,\mu^3}$, k and α arbitraries and $\beta = 3$.

5. For $n = 1$, $a = \frac{\alpha^{-m}\,(m+1)\,(m+2)\,\lambda}{2\,\mu}$, $b = \frac{m^2\,\lambda}{4\,\mu^3}$, $k = 0$, α arbitrary and $\beta = \frac{2}{m}$.

6. For $m = n-1$, $a = \frac{2\,\alpha^{1-n}\,n^2\,\lambda}{\mu\,(n+1)}$, $b = \frac{\alpha^{1-n}\,(1-n)^2\,\lambda}{2\,\mu^3\,n\,(1+n)}$, $k = 0$, α arbitrary and $\beta = \frac{2}{n-1}$.

EQUATION (6). We can see that equation (6) admits for $m = \frac{n-1}{3}$ one group corresponding to the operator $\mathbf{w}_1 = z\partial_z + \frac{h}{m}\partial_h$. Transformation

$$z = t, \quad u = x^{\frac{3}{n-1}}\, h(z) \tag{10}$$

reduces equation (6) to

$$-\frac{3\,n\,(1+2\,n)\,(2+n)}{(1-n)^3} h^n + \frac{3\,a}{(n-1)}\, h^{\frac{n+2}{3}} + h' = 0.$$

For $n = \frac{1}{2}$ and $m = -\frac{1}{2}$ equation (6) admits one group corresponding to the operator $\mathbf{w}_2 = \partial_z$. We find that for the new variables (invariants of the first prolongation of $\mathbf{w}_2$) $y = h$ and $\varphi = h'$ equation (6) takes the following form

$$b\,n\left(\varphi^2\,\varphi'' + \varphi^3\,(\varphi')^2\right) y^{n+2} - a\,\varphi\, y^{n+2}$$

$$+3\,b\,(n-1)\,n\,\varphi^2\,\varphi'\,y^{n+1}$$

$$+b\,(n-2)\,(n-1)\,n\,\varphi^3\,y^n + 2\,y^{2\,n+3} = 0.$$

3. TRAVELING WAVE AND SOURCE SOLUTIONS FOR THE $K(M,N)$ EQUATION

Traveling wave solutions. From theorem 1 by making

$$u(x,t) = h(\mu x - \lambda t) \tag{11}$$

we obtain the following exact solutions for the $K(m,n)$ equation (2).

From theorem 1 with m, n, a and b given in the different cases:

1.1.

$$u(x,t) = \frac{-3k}{\lambda}\mathrm{sn}^4(\mu x - \lambda t, k).$$

1.2.

$$u(x,t) = -\frac{3\,k}{(k-1)\,\lambda}\mathrm{cn}^4(\mu x - \lambda t, k).$$

1.3.

$$u(x,t) = -\frac{3\,(k-1)}{k\,\lambda}\mathrm{cn}^{-4}(\mu x - \lambda t, k).$$

2.1.

$$u(x,t) = \alpha\,\mathrm{sn}^{-\frac{2}{1-n}}(\mu x - \lambda t, -1).$$

2.2.

$$u(x,t) = \alpha\,\mathrm{sn}^{-\frac{2}{3-n}}(\mu x - \lambda t, -1).$$

3.

$$u(x,t) = \alpha\,\mathrm{sn}(\mu x - \lambda t, k).$$

4.

$$u(x,t) = \alpha\,\mathrm{sn}^3(\mu x - \lambda t, k).$$

5.

$$u(x,t) = \alpha\,\mathrm{sech}^{\frac{2}{m}}(\mu x - \lambda t).$$

6.

$$u(x,t) = \alpha\,\sin^{\frac{2}{n-1}}(\mu x - \lambda t).$$

Source solutions. If $n = 3 + m$ the similarity solution has the form

$$u(x,t) = \frac{1}{t^{\frac{2}{m-5}}} h\left(\frac{x}{t^{\frac{2}{m-5}}}\right),$$

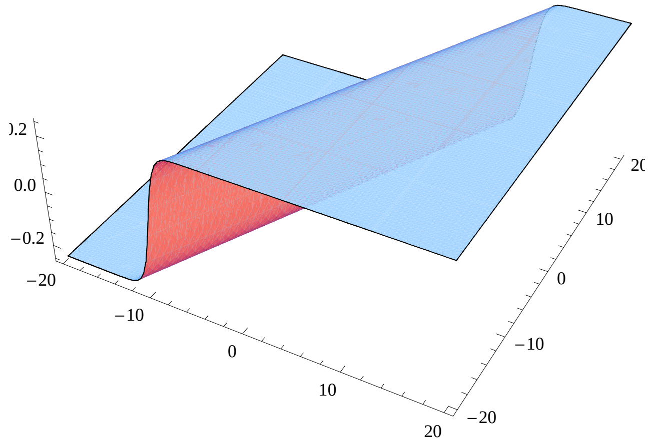

Fig. 1. Solution (12) for $\mu = 1$ and $\lambda = -\frac{1}{2}$

thus, if $m > 5$ it is clear that $u(x,t) \to \delta(x)$ as $t \to 0$ and the similarity solution is a source solution.

4. SOME EXAMPLES

Below we get some solutions with physical interest:

From **3** we obtain that

$$u(x,t) = \frac{1}{4}\tanh(\mu x - \lambda t) \tag{12}$$

is a solution of equation (2) with $n = 1$, $m = 2$, $a = \frac{48\lambda}{\mu}$ and $b = -\frac{\lambda}{2\mu^3}$. In Figure 1 we plot solution (12) with $\mu = 1$ and $\lambda = -\frac{1}{2}$ which describes a kink solution.

From **4** we obtain that

$$u(x,t) = \tanh^3(\mu x - \lambda t) \tag{13}$$

is a solution of equation (2) with $n = \frac{1}{3}$, $m = -\frac{2}{3}$, $a = \frac{\lambda}{3\mu}$ and $b = \frac{\lambda}{2\mu^3}$. In Figure 2 we plot solution (13) with $\mu = 1$ and $\lambda = \frac{1}{2}$ which describes an anti-kink solution.

5. CONCLUSIONS

We have studied the one-dimensional $K(m,n)$ equation (2), by making use of the theory of symmetry reductions in differential equations, and by focusing our attention in those aspects with physical interest. In this way, besides obtaining a complete classification of the Lie symmetries admitted by (2) depending on the values of the parameters a, b, m, n, we have also found:

- $K(m,n)$ equation (2) for $m > 5$ admits source solutions.
- $K(m,n)$ equation (2) has traveling wave solutions. Making use of these reduction transformations, we obtain some particular traveling-wave solutions for the cases above: kink and anti-kink.

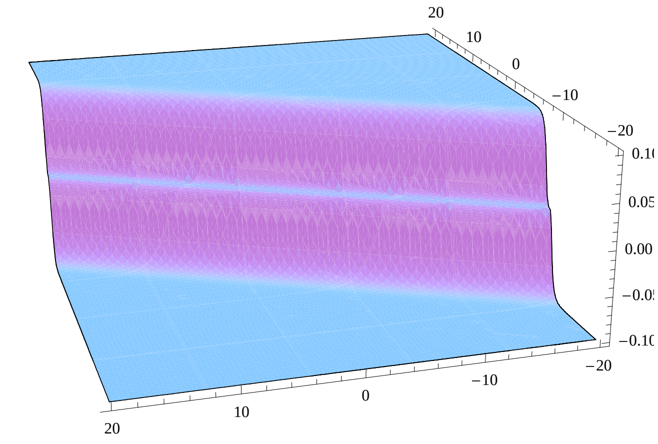

Fig. 2. Solution (13) for $\mu = 1$ and $\lambda = \frac{1}{2}$

REFERENCES

M.J. Ablowitz and H. Segur. Solitons and the Inverse Scattering Transform. *SIAM Studies in Applied Mathematics*, 4, Pennsylvania, 1981.

G.W. Bluman and S. Kumei. Symmetries and differential equations. Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1989.

M.S. Bruzón, M.L. Gandarias and J.C. Camacho. Classical and nonclassical symmetries for a Kuramoto-Sivashinsky equation with dispersive effects. *Math. Meth. Appl. Sci.*, 30 2091–2100, 2007.

M.S. Bruzón, M.L. Gandarias and J.C. Camacho. Symmetry for a Family of BBM Equations. *J. Nonl. Math. Phys*, 15 81–90, 2008.

M.S. Bruzón. Exact Solutions for a Generalized Boussinesq Equation. *Theor. Math. Phys*, 160:894-904, 2009.

M.S. Bruzón and M.L. Gandarias. Travelling wave solutions for a generalized double dispersion equation. *Nonlinear Analysis*, 71 e2109-e2117, 2009.

P.A. Clarkson and M.D. Kruskal. New similarity reductions of the Boussinesq equation. *J. Math. Phys.*, 30: 2201–2213, 1989.

A. Das. Integrable Models. *World Scientific Lecture Notes in Physics*, 30, World Scientific, New Jersey, 1989.

M.L. Gandarias. Nonclassical potential symmetries of the Burgers equation. *Symmetry in Nonlinear Mathematical Physics. National Academy of Sciences of Ukraine and Institute of Mathematics*. Kiev, 130-137, 1997.

M.L. Gandarias and M.S. Bruzón. Solutions through nonclassical potential symmetries for a generalized inhomogeneous nonlinear diffusion equation. *Math. Meth. Appl. Sci.*, 31 753-767, 2008.

N.A. Kudryashov. Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos, Solitons and Fractals*, 24 1217–1231, 2005.

N.A. Kudryashov and N.B. Loguinova. Extended simplest equation method for nonlinear differential equations. *Appl. Math. and Comput*, 205 396–402, 2008.

R.M. Miura. The Korteweg-de Vries equation: a survey of results. *SIAM Rev.*, 18:412-459, 1976.

A.C. Newell. The interrelation between Backlund transformations and the inverse scattering transform, Backlund Transformations, the Inverse Scattering Method, Solitons, and Their Applications. (Workshop Contact Transformations, Vanderbilt Univ.,Nashville, Tenn, 1974) (RM Miura, ed.). *Lecture Notes in Math.*, 515, Springer, Berlin, 227-240, 1976.

P. Rosenau and J.M. Hyman. Compactons: solitons with finite wavelengths. *Phys. Rev. Lett.*, 70:564–567, 1993.

P. Rosenau. Nonlinear dispersion and compact structures. *Phys. Rev. Lett.*, 73:1737–1741, 1994.

P. Rosenau. On nonanalytic solitary waves formed by a nonlinear dispersion. *Phys. Lett. A*, 230 305–318, 1997.

P. Rosenau. On a class of nonlinear dispersive-dissipative interactions. *Phys. D*, 230(5-6) 535–546, 1998.