Unsteady MHD flow past a stretching sheet due to a heat source/sink

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Abstract: This paper deals with the unsteady heat transfer effects due to a sudden introduction of heat source/sink on a steady viscous boundary layer MHD flow and heat transfer over a linearly stretching sheet subjected to a constant temperature. Governing boundary layer equations have been solved by an implicit finite difference method. Numerical results show that the steady state is reached quickly for a heat sink or for a large Prandtl number. The time to reach steady state increases under magnetic field. Upto a critical value of the strength of heat source, steady solution exists.

Keywords: Unsteady boundary layer; Stretching sheet; Magnetic field; Heat source/sink; Finite difference method

1. INTRODUCTION

Steady heat transfer analysis in the presence of heat source has been receiving wide attention among the researchers due to its applications in polymer extrusion process, metallurgical process, drawing of artificial fibres etc. Recently, Liu (2005) studied heat and mass transfer in a MHD flow past a stretching sheet including the chemically reactive species of order one and internal heat generation or absorption. Xu (2005) studied the free convective heat transfer characteristics in an electrically conducting fluid near an isothermal sheet with internal heat generation or absorption.

Viscous dissipation effect along with heat transfer in MHD viscoelastic fluid flow over a stretching sheet was studied by Abel and Mahesha (2008). Their study also includes the effect of variable thermal conductivity, non-uniform heat source and radiation. Khan (2006) studied the effect of heat transfer on a viscoelastic fluid flow over a stretching sheet with heat source/sink, suction/blowing and radiation.

Pal and Talukdar (2010) studied the unsteady MHD heat and mass transfer along with heat source past a vertical permeable plate using a perturbation analysis, where the unsteadiness is caused by the time dependent surface temperature and concentration. Unsteady flow and heat transfer over an unsteady stretching sheet was studied by Ishak et al. (2009). Liu and Andersson (2008) have also studied the heat and flow transfer over an unsteady stretching sheet. Unsteady flow and heat transfer with viscous dissipation of a non-Newtonian fluid over an unsteady stretching sheet was considered by Chen (2006). Radiation effect on the unsteady flow and heat transfer over an unsteady stretching sheet was studied using fifth order Runge-Kutta-Fehlberg integration scheme by El-Aziz (2009). Chebyshev finite difference method was used by Tsai et al. (2008) to study the unsteady flow and heat transfer over an unsteady stretching surface with non-uniform heat source. Mukhopadhyay (2009) studied the unsteady heat and flow transfer along with radiation effect over an unsteady stretching sheet. It must be noted that the unsteadiness in Ishak et al. (2009)-Tsai et al. (2008) are due to time-dependent stretching rate and temperature of the sheet. In the present paper, the transient changes in the temperature field is due to the sudden introduction of the heat source/sink on the steady MHD boundary layer flow and heat transfer past a linearly stretching isothermal sheet. The effects of heat source/sink, magnetic field and Prandtl number on the temperature field are analysed.

2. MATHEMATICAL FORMULATION

2.1 Initial state ($t' \leq 0$)

Consider a steady two dimensional laminar boundary layer flow and heat transfer of an incompressible electrically conducting Newtonian fluid past a linearly stretching isothermal sheet under a transverse magnetic field of strength B_0. The sheet issues from a thin slit at $x' = 0, y' = 0$, where x'-axis is along the horizontal direction of flow, y'-axis is normal(vertically upwards) to the flow; u' and v' are the horizontal and vertical components of velocity along x' and y'-directions respectively. The stretching speed is proportional to the distance from the origin along the x'-direction, and the stretching rate is $\beta(>0)$. The sheet is assumed to be at constant temperature T'_w, far away the constant ambient fluid temperature is T'_∞ and T'_0 is the temperature of the fluid. Under these assumptions, the governing steady state boundary layer equations for the time $t^{'} \leq 0$ are given by Char (1994) as,

⋆

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0, \quad (1)$$

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = \nu\frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_0}{\rho}u', \quad (2)$$

$$u'\frac{\partial T_0'}{\partial x'} + v'\frac{\partial T_0'}{\partial y'} = \alpha\frac{\partial^2 T_0'}{\partial y'^2}, \quad (3)$$

subjected to the boundary conditions,

$$\begin{aligned} y' = 0 &: u' = \beta x', v' = 0, T_0' = T_w', \text{ for } x' \geq 0, \\ y' \to \infty &: u' \to 0, T_0' \to T_\infty', \text{ for } x' \geq 0, \end{aligned} \quad (4)$$

where ρ is the density, μ is the viscosity, ν is the kinematic viscosity, σ is the electric conductivity and α is the thermal diffusivity of the fluid.

Defining the dimensionless variables and parameters,

$$x = x'\sqrt{\frac{\beta}{\nu}}, \quad y = y'\sqrt{\frac{\beta}{\nu}}, \quad u = \frac{u'}{\sqrt{\nu\beta}}, \quad v = \frac{v'}{\sqrt{\nu\beta}}, \quad (5)$$

$$T_0 = \frac{T_0' - T_\infty'}{T_w' - T_\infty'}, \quad M = \frac{\sigma B_0}{\rho\beta}, \quad Pr = \frac{\nu}{\alpha}, \quad (6)$$

the equations governing the initial state take the dimensionless form,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (7)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - Mu, \quad (8)$$

$$u\frac{\partial T_0}{\partial x} + v\frac{\partial T_0}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T_0}{\partial y^2}, \quad (9)$$

subjected to the boundary conditions,

$$\begin{aligned} y = 0 &: u = x, v = 0, T_0 = 1, \text{ for } x \geq 0, \\ y \to \infty &: u \to 0, T_0 \to 0, \text{ for } x \geq 0. \end{aligned} \quad (10)$$

The Eqs. (7)-(10) admit a closed form solution, which is given by,

$$u = xe^{-\sqrt{1+M}y}, \; v = \frac{(e^{-\sqrt{1+M}y} - 1)}{\sqrt{1+M}}, \quad (11)$$

$$T_0(y) = e^{-\frac{Pr}{\sqrt{1+M}}y}\frac{F\left(a, a+1, -ae^{-\sqrt{1+M}y}\right)}{F\left(a, a+1, -a\right)}, \quad (12)$$

where $a = \frac{Pr}{1+M}$ and $F(a_0, a_1, z)$ is the Kummer's function (Char (1994)).

Substituting $\eta = y\sqrt{1+M}$ in Eq. (12) one can get the solution obtained by Char (1994), and Kumari and Nath (2009).

2.2 Transient heat transfer ($t' > 0$)

Assuming that the flow and temperature field when $t' \leq 0$, are given by Eqs. (11) and (12), a heat source/sink of constant strength Q' is introduced at time $t' = 0$ in the fluid flow and maintained for $t' > 0$. When $t' > 0$, the equation governing the transient heat transfer is given by,

$$\frac{\partial T'}{\partial t'} + u'\frac{\partial T'}{\partial x'} + v'\frac{\partial T'}{\partial y'} = \alpha\frac{\partial^2 T'}{\partial y'^2} + Q'(T' - T_\infty'), \quad (13)$$

subjected to the boundary conditions,

$$t' > 0: \quad T' = T_w' \text{ at } y' = 0, T' \to T_\infty' \text{ as } y' \to \infty. \quad (14)$$

Introducing the dimensionless quantities (along with Eq. (5) and (6)),

$$t = \beta t', T(x, y, t) = \frac{T' - T_\infty'}{T_w' - T_\infty'}, Q = \frac{Q'}{\beta}, \quad (15)$$

the dimensionless transient temperature field due to the sudden introduction of heat source/sink is given by,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial y^2} + QT, \; (t > 0) \quad (16)$$

$$t > 0: \quad T = 1 \text{ at } y = 0, T \to 0 \text{ as } y \to \infty, \quad (17)$$

$$t \leq 0: \quad T = T_0(y), \text{ for } x \geq 0, \, y \geq 0, \quad (18)$$

where the flow field is governed by Eq. (11) for $t > 0$ also and $T_0(y)$ is given by Eq. (12).

At steady state(when $t \to \infty$), Eqs. (16) and (17) admit a closed form solution, which is given by,

$$T = e^{-(\frac{K+a}{2})\sqrt{1+M}y}\frac{F\left(\frac{K+a}{2}, K+1, -ae^{-\sqrt{1+M}y}\right)}{F\left(\frac{K+a}{2}, K+1, -a\right)}, \quad (19)$$

where $K = \sqrt{a^2 - 4aQ}$.

The form of K imposes a restriction on Q such that $Q \leq Q_c$ where $Q_c = \frac{Pr}{4(1+M)}$.

The local skin friction defined by, $\tau_x' = -\mu\left.\frac{\partial u'}{\partial y'}\right|_{y'=0}$, takes the dimensionless form,

$$\tau_x = -u_y(x, 0) = x\sqrt{M+1}. \quad (20)$$

The dimensionless form of the local Nusselt number is given by,

$$Nu_x = -x\left(\frac{1}{T}\frac{\partial T}{\partial y}\right)_{y=0} = -x\left(\frac{\partial T}{\partial y}\right)_{y=0}. \quad (21)$$

The dimensionless average Nusselt number averaged over $0 \leq x \leq 1$, is given by

$$\overline{Nu} = -\int_0^1 \left(\frac{\partial T}{\partial y}\right)_{y=0} dx. \quad (22)$$

Computations reveal that, T is indepedent of x for all time (including $t > 0$). Hence,

$$\overline{Nu} = \frac{Nu_x}{x} = -\left.\frac{\partial T}{\partial y}\right|_{y=0}. \quad (23)$$

3. NUMERICAL SOLUTION

The numerical results for the unsteady state dimensionless temperature distribution is computed by solving Eqs. (16)-(18) using the implicit finite difference method of Crank-Nicholson type (Muthucumaraswamy and Ganesan (2000), Kumaran et al. (2010) and Kumaran et al. (2008)). For the present problem when the Prandtl number $Pr = 0.71$, the mesh size was taken as $\Delta x = 0.002$, $\Delta y = 0.0125$ and $\Delta t = 0.01$. The domain of computation was taken as $0 \leq x \leq 1$ and $0 \leq y \leq 35$. Whereas for $Pr = 7.0$, the domain of computation was taken as $0 \leq x \leq 1$ and $0 \leq y \leq 12$. The convergence criteria was set as $\left|T_{i,j}^{n+1} - T_{i,j}^n\right| \leq 1 \times 10^{-5}$, where i, j denote the mesh point along the x-axis, y-axis respectively and n denotes the number of iterations with respect to time.

4. RESULTS AND DISCUSSION

The effects of heat source/sink have been studied for various values of the parameters namely, the heat source ($Q > 0$)/heat sink ($Q < 0$) parameter, magnetic parameter M and Prandtl number Pr. The Fig. 1 describe the profiles of steady state ($t \to \infty$) excess temperature due to heat source/heat sink. From the Figs. 1(a) and 1(b) it is observed that the thermal boundary layer is thin for $Pr = 7.0$ whereas it is thick for $Pr = 0.71$. The temperature raises or drops if the the magnitude of the heat source/heat sink increases respectively. These effects, are more pronounced under magnetic field. From Table-1 it is seen that the values of the computed steady local Nusselt number at $x = 1$ are in good agreement with the values of the exact steady local Nusselt number obtained from Eq. (12).

The Figs. 2(a)-5(b) describe the evolution of the temperature profiles with respect to time. Increase in temperature and decrease in temperature is seen to occur for positive Q (heat source) and negative Q (heat sink) respectively. Tables-2 and 3 give the time to reach steady state for $Pr = 0.71$ and $Pr = 7.0$ for various values of Q. It is observed that for an increase in strength of the heat sink, the steady time decreases. The value Q_c represents the critical value given by $Q_c = Pr/(4(1 + M))$. The computations converges for $Q \leq Q_c$ only. The steady state time increases with an increase in Q, M and decreases with an increase in Pr. Steady state is reached very quickly for $Pr = 7.0$ when compared to $Pr = 0.71$. For $Q > Q_c$ the computation doesnot terminate justifying the fact that steady solution exists only for $Q \leq Q_c$. The excess average Nusselt number increases with M whereas it decreases with Q, as seen from Fig. 6.
The following important observations are made:

- A critical value Q_c of strength of heat source exists beyond which no steady solution exists.
- Reaching steady state is delayed under magnetic field.
- The steady state time decreases with an increase in Prandtl number Pr.
- Steady state is reached quickly for a heat sink whereas it is delayed for a heat source of same strength.
- The local Nusselt number decreases with an increase in M, Q and for a decrease in Pr.

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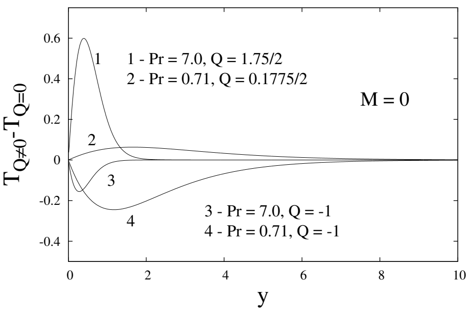
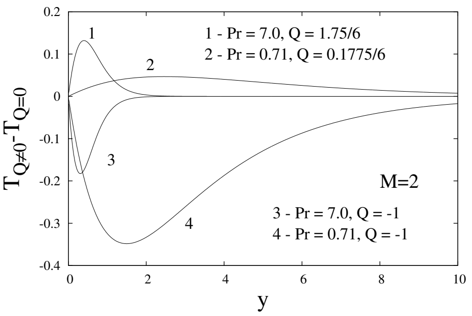
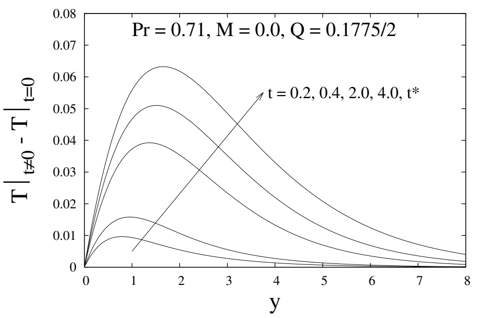


(a) $M = 0$

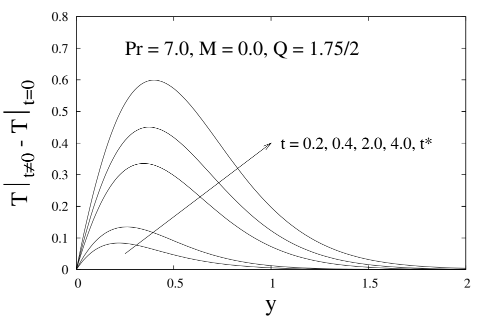


(b) $M = 2$

Fig. 1. Profiles of the steady state excess temperature due to heat source/sink (a) $M = 0$, (b) $M = 2$.

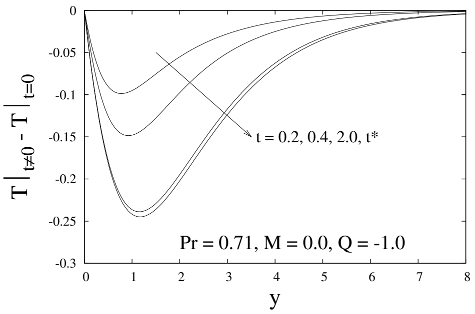


(a) $Pr = 0.71$

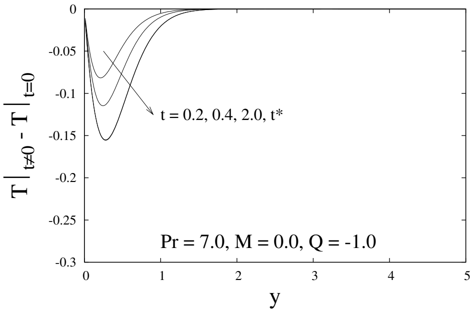


(b) $Pr = 7.0$

Fig. 2. Profiles of the transient excess temperature due to heat source for $M = 0$ (a) $Pr = 0.71$, (b) $Pr = 7.0$.

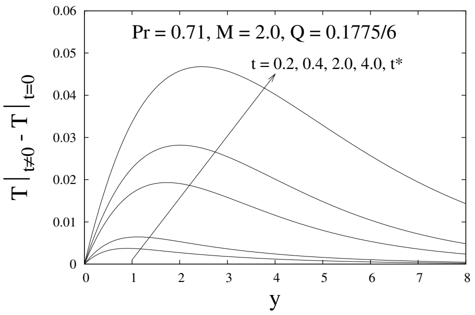


(a) $Pr = 0.71$

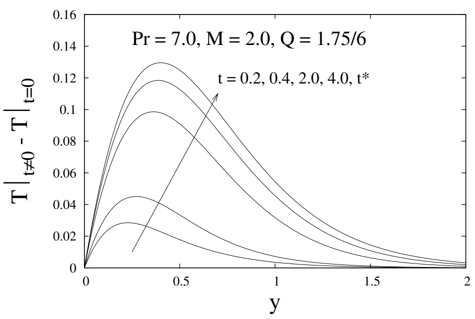


(b) $Pr = 7.0$

Fig. 3. Profiles of the transient excess temperature due to heat sink of strength $Q = -1$ and $M = 0$ (a) $Pr = 0.71$, (b) $Pr = 7.0$.

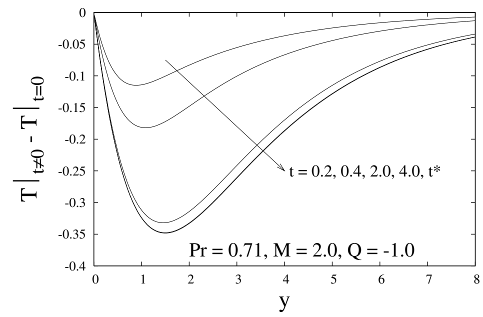


(a) $Pr = 0.71$

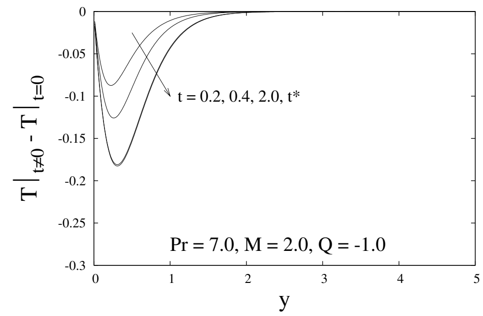


(b) $Pr = 7.0$

Fig. 4. Profiles of the transient excess temperature due to heat source for $M = 2$ (a) $Pr = 0.71$, (b) $Pr = 7.0$.

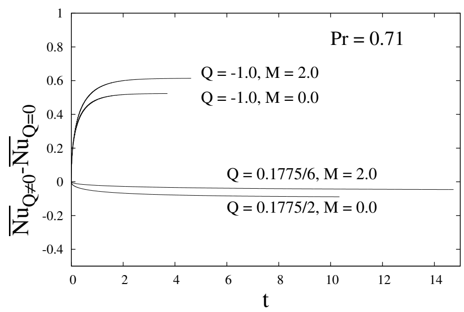


(a) $Pr = 0.71$

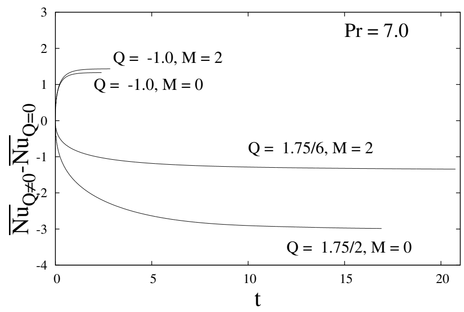


(b) $Pr = 7.0$

Fig. 5. Profiles of the transient excess temperature due to heat sink of strength $Q = -1$ for $M = 2$ (a) $Pr = 0.71$, (b) $Pr = 7.0$.



(a) $Pr = 0.71$



(b) $Pr = 7.0$

Fig. 6. Profiles of the excess average Nusselt number(averaged over $0 \leq x \leq 1$) due to heat sink/source (a) $Pr = 0.71$, (b) $Pr = 7.0$.

Table 1. Values of the steady local Nu_x at $x = 1$ where $Q_c = \frac{Pr}{4+4M}$

Pr	M	Q	(Exact) Nu_x	(Computed) Nu_x
0.71	0	-1	-0.9826	-0.9822
		0.0	-0.4585	-
		$Q_c/2$	-0.3657	-0.3699
	2	-1	-0.9521	-0.9517
		0	-0.3379	-
		$Q_c/2$	-0.2850	-0.2919
7.0	0	-1	-3.2271	-3.2257
		0.0	-1.8954	-
		$Q_c/2$	1.1703	1.1625
	2	-1	-3.1714	-3.17
		0	-1.7352	-
		$Q_c/2$	-1.0058	-1.0136

Table 2. The values of t^*, the time to reach steady state for $Pr = 0.71$

		Q						
		-1.0	-0.5	0	0.04	$Q_c/4$	$Q_c/2$	Q_c
	0	3.71	5.02	6.48	5.86	6.00	10.34	40.57
M	1	4.30	6.25	8.52	11.79	7.08	13.04	40.43
	2	4.62	6.96	9.36	20.62	7.43	14.74	42.53

Table 3. The values of t^*, the time to reach steady state for $Pr = 7.0$

		Q					
		-1.0	-0.5	0	0.04	$Q_c/4$	$Q_c/2$
	0	2.40	3.78	3.14	3.16	5.84	16.93
M	1	2.65	3.15	3.61	3.62	5.01	9.26
	2	2.85	3.47	4.01	4.01	4.95	8.51

Nomenclature

B_0	strength of the magnetic field, $kgs^{-2}A^{-1}$
M	dimensionless magnetic parameter, Eq. (6)
n	number of iterations
Nu_x	local Nusselt number
$\overline{Nu}$	average Nusselt number (over $0 \leq x \leq 1$)
Pr	Prandtl number
Q'	Heat source/sink strength, Wm^{-2}
t'	time, s
T'_w	Temperature at the sheet, K
T'_∞	Temperature of the ambient fluid, K
u'	velocity component along the sheet, ms^{-1}
v'	velocity component normal to the sheet, ms^{-1}
x'	coordinate along the sheet, m
y'	coordinate normal to the sheet, m
t	dimensionless time
T_0	dimensionless steady temperature, Eq. (12)
T	dimensionless transient temperature, Eq. (16)
u	dimensionless speed along the sheet
v	dimensionless speed normal to the sheet
x	dimensionless coordinate along the sheet
y	dimensionless coordinate normal to the sheet
Greek symbol	
α	thermal diffusivity, m^2s^{-1}
β	stretching rate, s^{-1}
τ_x	local skin friction, $kgs^{-2}m^{-1}$
Δt	dimensionless step size with respect to time
Δx	dimensionless step size along x
Δy	dimensionless step size along y
η	$y\sqrt{1+M}$
τ_x	dimensionless local skin friction
μ	dynamic viscosity, $kgm^{-1}s^{-1}$
ν	kinematic viscosity, m^2s^{-1}
ρ	density, kgm^{-3}
σ	electric conductivity, $kg^{-1}m^3A^2$
Subscript	
w	property at the wall
∞	free stream condition