Uncertainty in a mixed duopoly with quadratic costs

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Abstract: In this paper, we consider a mixed market with uncertain demand, involving one private firm and one public firm with quadratic costs. The model is a two-stage game in which players choose to make their output decisions either in stage 1 or stage 2. We assume that the demand is unknown until the end of the first stage. We compute the output levels at equilibrium in each possible role. We also determine ex-ante and ex-post firms' payoff functions.

Keywords: Game Theory, optimization problems, uncertainty, Cournot model, leadership.

1. INTRODUCTION

Since the 1980s we have observed a worldwide wave of privatization. Nevertheless, many public firms still exist, and many of them compete with private firms in private goods markets in developed, developing and former communist countries. Competition between public firms exists or still exist, in a range of industries, including airlines, railways, telecommunications, natural gas, electricity, steel and overnight delivery, as well as services such as banking, home loans, health care, life insurance, hospitals, broadcasting and education. Many works on mixed oligopoly consider that public and private firms set output either simultaneously or sequentially giving rise to a Cournot or Stackelberg structure. Furthermore, the role of each firm is usually given exogenously. In many economic situations, however, it is often more reasonable to assume that firms choose not only what actions to take, but also when to take them. Endogenous timing might be important since an alternative order of moves often gives rise to different results (see, for example, Dowrick (1986); Gal-Or (1985); Hamilton and Slutsky (1990) and Matsumura (2003)).

In this paper, we consider a two-stage mixed duopoly model in which players choose to make their output decisions either in stage 1 or stage 2. We consider that the demand is unknown until the end of the first stage. In a non-stochastic model, it is well-known that the leader has no incentive to deviate from the committed output, since follower's reaction is incorporated in setting that output (see Matsumura (2003); Pal (1998)). In the stochastic model, however, the follower firm may have higher (ex-post) profits than the leader. We follow Anam et al. (2007), by considering a more general inverse demand function. We should mention that issues related to those of this paper have been studied by Ferreira et al. (2008) and Ferreira (2009).

2. THE MODEL

We consider a two-stage model in which firms choose to make their output decision either in stage 1 or stage 2. Firm F_1 is a social-welfare-maximizing firm (i.e a public firm), and firm F_2 is a profit-maximizing firm (i.e a private firm). The inverse demand function is given by

$$P = a - bQ + \Delta = 1 - b(q_1 + q_2) + \Delta,$$

where P is the price, Q is total output, q_1 is the output of the public firm F_1 and q_2 is the output of the private firm F_2; $a > 0$ is a demand parameter large enough for the equilibrium quantities to be always positive (i.e., $a > b(q_1 + q_2) - \Delta$); $b \geq 1$ is the slope parameter. The demand uncertainty is represented by a random variable Δ with expectation $E(\Delta) = 0$ and variance $V(\Delta) = \sigma^2 > 0$. The value of Δ is unknown to both players in stage 1, but it becomes known at the beginning of stage 2. We assume that before the output game begins, the firm determines simultaneously whether to move early (E) and produce output in stage 1 or to follow late (L) and produce output in stage 2. Since the random variable Δ will not be revealed until the end of the first stage, the early mover would have to make the output decision before the random variable becomes known. The late mover, however, makes his decision after the complete resolution of uncertainty. Depending on the timing of their moves, we can consider four possible combinations: (i) Both firms choose to move late, denoted by (L, L); (ii) Both firms choose to move early, denoted by (E, E); (iii) Firm F_1 moves early and firm F_2 moves late, denoted by (E, L); and (iv) Firm F_1 moves late and firm F_2 moves early, denoted by (L, E). In cases (i) and (ii), firm's actions are the same; so the model results in a Cournot model. In cases (iii) and (iv), firm's actions are different; so the model results in a Stackelberg model.

We assume that both firms have identical technologies, represented by the cost function $C_i(q_i) = \frac{1}{2}q_i^2$, with $i \in$

$\{1,2\}$ (see Anam et. al Anam et al. (2007)). The profit of firm F_i is given by

$$\pi_i(q_1,q_2) = (a - bq_1 - bq_2 + \Delta)q_1 - \frac{1}{2}q_i^2.$$

Public firm F_1 maximizes social welfare W which is defined as the sum of producer surplus and consumer surplus:

$$\begin{aligned} W &= \int_0^Q p(x)dx - pQ + \pi_1(q_1,q_2) + \pi_2(q_1,q_2) \\ &= \int_0^Q p(x)dx - C_1(q_1) - C_2(q_2) \\ &= \frac{b}{2}Q^2 + (a - bQ + \Delta)Q - \frac{1}{2}q_1^2 - \frac{1}{2}q_2^2. \end{aligned} \quad (1)$$

We will analyse the four possible cases separately.

2.1 Case 1: (L,L)

In this case, both firms move late; so, firms decide their outputs after Δ is revealed to both firms at the end of stage 1. Maximizing firms' objective functions, we get the following equilibrium output levels:

$$q_1(\Delta) = \frac{(b+1)(a+\Delta)}{b^2+3b+1},$$
$$q_2(\Delta) = \frac{a+\Delta}{b^2+3b+1}.$$

Therefore, social welfare is given by

$$\begin{aligned} A_{L,L} = W = &\frac{\left(b^3+5b^2+8b+2\right)\sigma^2}{2\left(b^2+3b+1\right)^2} + \\ &+\frac{\left(b^3+5b^2+8b+2\right)a^2}{2\left(b^2+3b+1\right)^2}, \end{aligned} \quad (2)$$

and firm F_2's profit is given by

$$B_{L,L} = \pi_2 = \frac{(2b+1)\sigma^2}{2\left(b^2+3b+1\right)^2} + \frac{(2b+1)a^2}{2\left(b^2+3b+1\right)^2}. \quad (3)$$

We note that, in the presence of uncertainty ($\sigma^2 > 0$), taking the output decision after the resolution of the random variable enhances firms' payoffs since firms are now able to make more well-informed decisions. The benefit of making a well-informed decision is captured by the first term in equations (2) and (3). This is called the *option value* effect. This option value increases with the degree of uncertainty. Clearly, the option value effect ceases to prevail under certainty. In this case, waiting does not carry any information value.

2.2 Case 2: (E,E)

In this case, both firms move early; so, firms decide their outputs before Δ becomes known. Maximizing firms' objective functions, we get the following equilibrium output levels:

$$q_1 = \frac{(b+1)a}{b^2+3b+1},$$
$$q_2 = \frac{a}{b^2+3b+1}.$$

Therefore, the (ex-ante) expected social welfare is given by

$$A_{E,E} = E(W) = \frac{\left(b^3+5b^2+8b+2\right)a^2}{2\left(b^2+3b+1\right)^2}, \quad (4)$$

and the (ex-ante) expected firm F_2's profit is given by

$$B_{E,E} = E(\pi_2) = \frac{(2b+1)a^2}{2\left(b^2+3b+1\right)^2}. \quad (5)$$

We observe that if $\sigma^2 = 0$, then $A_{L,L} = A_{E,E}$ and $B_{L,L} = B_{E,E}$.

Furthermore, the ex-post social welfare is given by

$$\begin{aligned} W = &\frac{(b+1)\left(b^2+4b+1\right)\Delta^2 + 2\left(b^3+5b^2+7b+2\right)a\Delta}{2\left(b^2+3b+1\right)^2} + \\ &+\frac{\left(b^3+5b^2+8b+2\right)a^2}{2\left(b^2+3b+1\right)^2}, \end{aligned} \quad (6)$$

and the ex-post firm F_2's profit is given by

$$\pi_2 = \frac{(b+1)\left((3b+1)\Delta^2 + 2(2b+1)a\Delta + (b+1)a^2\right)}{2\left(b^2+3b+1\right)^2}. \quad (7)$$

2.3 Case 3: (E,L)

In this case, firm F_1 moves early and firm F_2 moves late; so, the public firm acts as a Stackelberg leader, while the private firm is a follower. We determine the subgame perfect Nash equilibrium by backwards induction. Suppose that the public firm F_1 has chosen the output q_1 in the first stage.

Maximizing firm F_2's profit function, we get

$$q_2(q_1,\Delta) = \frac{a - bq_1 + \Delta}{2b+1}.$$

Now, maximizing expected social welfare

$$E(W(q_1, q_2(q_1,\Delta))),$$

knowing the above quantity $q_2(q_1,\Delta)$, we get

$$q_1 = \frac{\left(b^2+3b+1\right)a}{b^3+7b^2+5b+1}.$$

By substitution, we obtain

$$q_2 = \frac{(2b+1)a}{b^3+7b^2+5b+1} + \frac{\Delta}{2b+1}.$$

Therefore, the (ex-ante) expected social welfare is given by

$$A_{E,L} = E(W) = \frac{(3b+1)\sigma^2}{2(2b+1)^2} + \frac{\left(b^2+6b+2\right)a^2}{2\left(b^3+7b^2+5b+1\right)}, \quad (8)$$

and the (ex-ante) expected firm F_2's profit is given by

$$B_{E,L} = E(\pi_2) = \frac{\sigma^2}{2(2b+1)} + \frac{(2b+1)^3a^2}{2\left(b^3+7b^2+5b+1\right)^2}. \quad (9)$$

Furthermore, the ex-post social welfare is given by

$$W=\frac{(3b+1)\Delta^2}{2\,(2b+1)^2}+\frac{\left(2b^3+13b^2+10b+2\right)a\Delta}{(2b+1)\left(b^3+7b^2+5b+1\right)}+$$
$$+\frac{\left(b^2+6b+2\right)a^2}{2\left(b^3+7b^2+5b+1\right)^2}, \tag{10}$$

and the ex-post firm F_2's profit is given by

$$\pi_2=\frac{\Delta^2}{2(2b+1)}+(2b+1)a\Delta+\frac{(2b+1)^3a^2}{2\left(b^3+7b^2+5b+1\right)^2}. \tag{11}$$

2.4 Case 4: (L,E)

In this case, firm F_1 moves late and firm F_2 moves early; so, the public firm acts as a follower, while the private firm is a Stackelberg leader. We determine the subgame perfect Nash equilibrium by backwards induction. Suppose that the private firm F_2 has chosen the output q_2 in the first stage.

Maximizing social welfare, we get

$$q_1(q_2,\Delta)=\frac{a-bq_2+\Delta}{b+1}.$$

Now, maximizing expected firm F_2's profit

$$E(\pi_2(q_1(q_2,\Delta),q_2),$$

knowing the above quantity $q_1(q_2,\Delta)$, we get

$$q_2=\frac{a}{3b+1}.$$

By substitution, we obtain

$$q_1=\frac{(2b+1)a}{(b+1)(3b+1)}+\frac{\Delta}{b+1}.$$

Therefore, the (ex-ante) expected social welfare is given by

$$A_{L,E}=E(W)=\frac{\sigma^2}{2(b+1)}+\frac{\left(9b^2+10b+2\right)a^2}{2(b+1)(3b+1)^2}, \tag{12}$$

and the (ex-ante) expected firm F_2's profit is given by

$$B_{L,E}=E(\pi_2)=\frac{a^2}{2(b+1)(3b+1)}. \tag{13}$$

Furthermore, the ex-post social welfare is given by

$$W=\frac{\Delta^2}{2(b+1)}+\frac{(3b+2)a\Delta}{(b+1)(3b+1)}+$$
$$+\frac{\left(9b^2+10b+2\right)a^2}{2(b+1)(3b+1)^2}, \tag{14}$$

and the ex-post firm F_2's profit is given by

$$\pi_2=\frac{a^2+2a\Delta}{2(b+1)(3b+1)}. \tag{15}$$

3. CONCLUSIONS

In this paper, we studied a model in which timing and output games are played between a public and a private firm in a market with demand uncertainty. We computed the output levels at equilibrium in each possible role. We also determined ex-ante and ex-post firms' payoff functions.

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